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OPTIMAL CONTROL OF A CLASS OF DEGENERATE NONLINEAR EVOLUTION EQUATIONS

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Abstract. We examine problems of optimal control of systems driven by a nonlinear, degenerate evolution equation. First we establish two existence results for two different types of integral cost criteria. Then we examine the sensitivity of the optimal value on variations of the data. Finally we present an example of a degenerate, nonlinear parabolic optimal control system.

1. Introduction. In this paper we examine optimal control problems governed by a nonlinear degenerate evolution equation. First we establish two existence results for two different integral cost functionals. Then we study the changes in the optimal value, as the data of the problem vary. Finally we present an example illustrating the applicability of our results.

The mathematical setting is the following. Let $T = [0, b]$, H a separable Hilbert space and X a subspace of H carrying the structure of a separable, reflexive Banach space, which embeds continuously and densely into H. Identifying H with its dual (pivot space), we have $X \hookrightarrow H \hookrightarrow X^*$, with all embeddings being continuous and dense. Such a triple of spaces is usually called in the literature "Gelfand triple" or "evolution triple" or "spaces in normal position". By $\|\cdot\|$ (resp. $|\cdot|$, $\|\cdot\|_*$) we will denote the norm of X (resp. of H, X^*), by $(\cdot\,,\cdot)$ the inner product in H and by $\langle\cdot\,,\cdot\rangle$ the duality brackets for the pair (X, X^*). The last two are compatible in the sense that $\langle\cdot\,,\cdot\rangle|_{X\times H} = (\cdot\,,\cdot)$. Also, let Y be a separable reflexive Banach space, modelling the control space. By $P_{\mathrm{wkc}}(Y)$ we will denote the set of nonempty, weakly compact and convex subsets of Y. A multifunction $U : T \to P_{\mathrm{wkc}}(Y)$ is said to be L_2-integrably bounded if and only if $U(\cdot)$ is measurable and $t \to |U(t)| = \sup\{\|u\| : u \in U(t)\} \in L^2_+$. Recall that $U(\cdot)$ is measurable if and only if for every $v \in Y$, $t \to d(v, U(t)) = \inf\{\|v - u\| : u \in U(t)\}$ is measurable.

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Finally let Z be a Banach space and let $\{A_n, A\}_{n\geq 1} \subseteq 2^Z \setminus \{\varnothing\}$. We set $s\text{-}\underline{\lim}\, A_n = \{z \in Z : z = s\text{-}\underline{\lim}\, z_n, z_n \in A_n, n \geq 1\}$ and $w\text{-}\overline{\lim}\, A_n = \{z \in Z : z = w\text{-}\overline{\lim}\, z_{n_k}, z_{n_k} \in A_{n_k}, n_1 < n_2 < \ldots < n_k < \ldots\}$, where s denotes the strong topology on Z and w the weak topology on Z. It is clear from these two definitions that $s\text{-}\underline{\lim}\, A_n \subseteq w\text{-}\overline{\lim}\, A_n$. We will say that the A_n's converge to A in the "Kuratowski-Mosco" sense, denoted by $A_n \xrightarrow{\text{K-M}} A$, if and only if $s\text{-}\underline{\lim}\, A_n = A = w\text{-}\overline{\lim}\, A_n$. Next let $f_n, f : Z \to \overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$. We set $\operatorname{epi} f_n = \{(z, \lambda) \in Z \times \mathbb{R} : f_n(z) \leq \lambda\}$ (the epigraph of $f_n(\cdot)$). Similarly we define $\operatorname{epi} f$. We say that the f_n's epi-converge to f, denoted by $f_n \xrightarrow{\tau} f$ if and only if $\operatorname{epi} f_n \xrightarrow{\text{K-M}} \operatorname{epi} f$. From Mosco [**5**] we know that this is equivalent to saying that for every subsequence $\{f_{n_k}\}_{k\geq 1}$, if $z_k \xrightarrow{s} z$, then $f(z) \leq \underline{\lim}\, f_{n_k}(z_k)$ and for every $z \in Z$ there exists a sequence $z_n \xrightarrow{w} z$ s.t. $\lim f(z_n) = f(z)$.

2. Existence theorems. The first optimal control problem that we will examine is the following:

$$\begin{aligned} &J_1(x,u) = \int_0^b L(t, x(t), u(t))\, dt \to \inf = m_1 \\ &\text{s.t.} \quad (d/dt)(Ex(t)) + A(t, x(t)) = (Bu)(t) \quad \text{a.e.} \\ &\qquad\ \ x(0) = x_0 \in H \\ &\qquad\ \ u(t) \in U(t) \quad \text{a.e.}; \quad u(\cdot) \text{ is measurable} \end{aligned} \tag{$*$}_1$$

We will need the following hypotheses on the data of $(*)_1$.

$H(A)$: $A : T \times T \to X^*$ is an operator s.t.

(1) $t \to A(t,x)$ is measurable,
(2) $x \to A(t,x)$ is hemicontinuous, monotone,
(3) $\langle A(t,x), x\rangle \geq c\|x\|^2$, $t \in T$, $c > 0$,
(4) $\|A(t,x)\|_* \leq a(t) + b\|x\|$ a.e. with $a(\cdot) \in L^2_+$, $b \geq 0$.

$H(E)$: $E \in \mathcal{L}(H)_+$ is self-adjoint.

$H(B)$: $B : L^2(H) \to L^2(H)$ is completely continuous.

$H(U)$: $U : T \to P_{\text{wkc}}(Y)$ is an L^2-integrably bounded multifunction.

$H(L)_1$: $L : T \times H \times Y \to \overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$ is a proper integrand (i.e. $L \not\equiv +\infty$) s.t.

(1) $L(\cdot,\cdot,\cdot)$ is measurable,
(2) $L(t,\cdot,\cdot)$ is convex and l.s.c.,
(3) $\phi(t) - M(|x| + \|u\|) \leq L(t,x,u)$ a.e. with $\phi(\cdot) \in L^1$, $M \geq 0$.

Since our cost integrand is $\overline{\mathbb{R}}$-valued, to avoid trivial situations, we need the following feasibility hypothesis:

H_a: There exists an admissible "state-control" pair (x,u) s.t. $J_1(x,u) < \infty$.

THEOREM 2.1. *If the hypotheses $H(A)$, $H(E)$, $H(B)$, $H(U)$, $H(L)_1$ and H_a hold, then the problem $(*)_1$ admits a solution.*

Proof. We will start by determining some a priori bounds for the trajectories of our system. So let $x(\cdot) \in L^2(X)$ be such a solution. Multiply the evolution equation with $x(s)$ and then integrate over $[0,t]$, $t \in T$. Using Theorem 2 od Brezis [**3**] and the integration by parts formula (see Zeidler [**10**, Proposition 23.23, p. 432]), we have

$$|E^{1/2}x(t)|^2 - |E^{1/2}x(0)|^2 = 2\int_0^t \langle dEx(s)/ds, x(s)\rangle\, ds$$

and so

$$|E^{1/2}x(t)|^2 + 2\int_0^t \langle A(s,x(s)), x(s)\rangle\, ds = |E^{1/2}x_0|^2 + 2\int_0^t ((Bu)(s), x(s))\, ds$$

$$\Longrightarrow |E^{1/2}x(t)|^2 + 2c\int_0^t \|x(s)\|^2\, ds \le |E^{1/2}x_0|^2 + 2\int_0^t \|(Bu)(s)\|_*\|x(s)\|\, ds. \quad (1)$$

Applying Cauchy's inequality with $\varepsilon > 0$, we get

$$\int_0^t \|(Bu)(s)\|_*\|x(s)\|\, ds \le \frac{\varepsilon}{2}\int_0^t \|(Bu)(s)\|_*^2\, ds + \frac{1}{2\varepsilon}\int_0^t \|x(s)\|^2\, ds. \quad (2)$$

Let $\varepsilon = 1/c$ and use inequality (2) in inequality (1). This way we get,

$$|E^{1/2}x(t)|^2 + c\int_0^t \|x(s)\|^2\, ds \le |E^{1/2}x_0|^2 + \frac{1}{c}\|B\|_{\mathcal{L}}^2\int_0^t |U(s)|^2\, ds \quad (3)$$

$$\Longrightarrow \quad \|x\|_{L^2(X)} \le M_1. \quad (4)$$

Also recall that $E^{1/2}x(\cdot) \in C(T,H)$ (see Brezis [**3**]). So from (3) above we also have that

$$\|E^{1/2}x(\cdot)\|_{C(T,H)} \le M_2. \quad (5)$$

Finally note that

$$\int_0^b \|dEx(t)/dt\|_*^2\, dt \le \int_0^b \|A(t,x(t))\|_*^2\, dt + \int_0^b |(Bu)(t)|^2\, dt$$

$$\le \int_0^b (2a(t)^2 + 2b^2\|x(t)\|^2)\, dt + \|B\|_{\mathcal{L}}^2\int_0^b |U(t)|^2\, dt. \quad (6)$$

Using bound (4) in inequality (6), we get

$$\|dEx(\cdot)/dt\|_{L^2(X^*)} \le 2\|a\|_2 + 2b^2M_1^2 + \|B\|_{\mathcal{L}}^2\|\,|U|\,\|_2^2$$

$$\Longrightarrow \quad \|dEx(\cdot)/dt\|_{L^2(X^*)} \le M_3. \quad (6')$$

Now that we have all the above a priori bounds, let $\{(x_n, u_n)\}_{n\ge1}$ be a minimizing sequence of admissible "state-control" pairs. Because of (4), hypothesis

$H(U)$ and the Eberlein-Smulian theorem, by passing to a subsequence if necessary, we may assume that $x_n \xrightarrow{w} x$ in $L^2(X)$ and $u_n \xrightarrow{w} u$ in $S_U^2 \subseteq L^2(Y)$. We will show that the limit pair (x,u) is admissible too. Recall that S_U^2 is convex, closed and bounded. So $u \in S_U^2$. Let $\eta(\cdot) \in L^2(X)$ s.t. $\dot\eta \in L^2(H)$ and $\eta(b)=0$ (note that our hypotheses on $\eta(\cdot)$, $\dot\eta(\cdot)$ imply that $\eta(\cdot) \in C(T,H)$ and so $\eta(b)=0$ makes sense). Then from Brezis [**3**], we know that for every $n>1$

$$\begin{aligned} &-\int_0^b (x_n(t), dE\eta(t)/dt)\,dt + \int_0^b \langle A(t,x_n(t)),\eta(t)\rangle\,dt \\ &= (E^{1/2}x_0, R^{1/2}\eta(0)) + \int_0^b ((Bu_n)(t),\eta(t))\,dt. \end{aligned} \tag{7}$$

Note that since $x_n \xrightarrow{w} x$ in $L^2(X)$ and $u_n \xrightarrow{w} u$ in $L^2(Y)$, we have

$$\int_0^b (x_n(t), dE\eta(t)/dt)\,dt \to \int_0^b (x(t), dE\eta(t)/dt)\,dt \tag{8}$$

$$\text{and}\quad \int_0^b ((Bu_n)(t),\eta(t))\,dt \to \int_0^b ((Bu)(t),\eta(t))\,dt. \tag{9}$$

Also note that $\|A(t,x_n(t))\|_* \le a(t)+b\|x_n(t)\|$ a.e. So if $\hat A : L^2(X) \to L^2(X^*)$ is the Nemitsky (superposition) operator corresponding to $A(\cdot,\cdot)$, then we get

$$\|\hat A(x_n)\|_{L^2(X^*)} \le \|a\|_2 + bM_1, \qquad n \ge 1.$$

So by passing to a further subsequence if necessary, we may assume that

$$\hat A x_n \xrightarrow{w} v \qquad \text{in } L^2(X^*). \tag{10}$$

Next for $m,n \ge 1$ we have

$$\begin{aligned} &\langle d(Ex_n(t)-Ex_m(t)), x_n(t)-x_m(t)\rangle \\ &+ \langle A(t,x_n(t)) - A(t,x_m(t)), x_n(t)-x_m(t)\rangle \\ &= ((Bu_n)(t)-(Bu_m)(t), x_n(t)-x_m(t)) \quad \text{a.e.} \\ \Longrightarrow\quad & |E^{1/2}x_n(t)-E^{1/2}x_m(t)|^2 + \int_0^t \langle A(s,x_n(s))-A(s,x_m(s)), x_n(s)-x_m(s)\rangle\,ds \\ &= \int_0^t ((Bu_n)(s)-(Bu_m)(s), x_n(s)-x_m(s))\,ds \\ \Longrightarrow\quad & |E^{1/2}x_n(t)-E^{1/2}x_m(t)|^2 \le (Bu_n - Bu_m, \chi_{[0,1]}(x_n-x_m))_{L^2(H)} \to 0 \end{aligned}$$

(since B is, by hypothesis $H(B)$, completely continuous).

Therefore for every $t \in T$, $\{E^{1/2}x_n(t)\}_{n\ge1}$ is strongly Cauchy in H. Let measurable function $y : T \to H$ be the limit function, i.e. $E^{1/2}x_n(t) \xrightarrow{s} y(t)$ in H for all $t \in T$. From (5) we know that $\|E^{1/2}x_n(t)\| \le M_2$ for all $n \ge 1$ and all

$t \in T$. So applying the dominated convergence theorem, we get $E^{1/2}x_n \xrightarrow{s} y$ in $L^2(H)$. On the other hand we already know that $E^{1/2}x_n \xrightarrow{w} E^{1/2}x$ in $L^2(H)$. Hence $y(t) = E^{1/2}x(t)$ a.e. and by modifying $y(\cdot)$ on a Lebesgue null set, we have equality everywhere.

Next for every $n \geq 1$, we have

$$\begin{aligned}
&\langle dEx_n(t)/dt, x_n(t) - x(t)\rangle + \langle A(t, x_n(t)), x_n(t) - x(t)\rangle \\
&= ((Bu_n)(t), x_n(t) - x(t)) \quad \text{a.e.} \\
\Longrightarrow \quad & \frac{1}{2}|E^{1/2}x_n(b) - E^{1/2}x(b)| + \int_0^b \langle dEx(t)/dt, x_n(t) - x(t)\rangle \, dt \\
&+ (\hat{A}(x_n), x_n - x)_{L^2(X^*), L^2(X)} = \int_0^b ((Bu_n)(t), x_n(t) - x(t)) \, dt.
\end{aligned}$$

Note that as $n \to \infty$

$$|E^{1/2}x_n(b) - E^{1/2}x(b)|/2 \to 0, \qquad \textstyle\int_0^b \langle dEx(t)/dt, x_n(t) - x(t)\rangle \, dt \to 0$$

and $\int_0^b ((Bu_n)(t), x_n(t) - x(t)) \, dt \to 0$ (since $B(\cdot)$ is completely continuous). Therefore we get $\lim(\hat{A}_n(x_n), x_n - x)_{L^2(X^*), L^2(X)} = 0$. But because of the hypothesis $H(A)$, $\hat{A} : L^2(X) \to L^2(X^*)$ is hemicontinuous, monotone and so it has property (M) (see Zeidler [**10**, pp. 583–584]). Therefore $\hat{A}x = v$; i.e. $\hat{A}x_n \xrightarrow{w} \hat{A}x$ in $L^2(X^*)$. Then, going back to (7) and using convergences (8) and (9), in the limit, we get

$$\begin{aligned}
&\int_0^b \langle x(t), dE\eta(t)/dt\rangle \, dt + \int_0^b \langle A(t, x(t)), \eta(t)\rangle \, dt \\
&= (E^{1/2}x_0, E^{1/2}\eta(0)) + \int_0^b ((Bu)(t), \eta(t)) \, dt. \qquad (11)
\end{aligned}$$

Also because of inequality (6′), we know that we may assume that $dEx_n/dt \xrightarrow{w} z$ in $L^2(X^*)$.

On the other hand, viewed as X^*-valued distributions, in the distributional sense $dEx_n/dt \to dEx/dt$. Hence $dEx/dt = z \in L^2(X^*)$. Since $x \in L^2(X)$, $dEx/dt \in L^2(X^*)$ and satisfies (11) above, from Brezis [**3**, Theorem 2, p. 31] we deduce that $E^{1/2}x(\cdot) \in C(T, H)$ and (x, u) is admissible. Finally note that $J_1(\cdot, \cdot)$ is convex and l.s.c. So we have $J_1(x, u) \leq \underline{\lim}\, J_1(x_n, u_n) = m_1$. Since (x, u) is an admissible pair, we have $J_1(x, u) = m_1$ and hence (x, u) is the desired optimal pair. Q.E.D.

We have a second existence result for a functional involving the degeneracy operator E. Specifically we consider the following optimal control problems:

$$\begin{aligned}
&J_2(x, u) = \int_0^b L(t, Ex(t), u(t)) \, dt \to \inf = m_2 \\
&\text{s.t.} \quad \dot{x}(t) + A(t, x(t)) = (Bu)(t) \quad \text{a.e.} \qquad (*)_2 \\
&\qquad x(0) = x_0 \\
&\qquad u(t) \in U(t) \quad \text{a.e.}; \quad u(\cdot) \text{ is measurable}
\end{aligned}$$

Now the hypothesis on the cost integrand is the following:

$H(L)_2$: $L : T \times H \times Y \to \overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$ is an integrand s.t.

(1) $L(\cdot,\cdot,\cdot)$ is measurable,

(2) for every $t \in T$, $L(t,\cdot,\cdot)$ is l.s.c. on $H \times Y$,

(3) for every $(t,x) \in T \times H$, $L(t,x,\cdot)$ is convex,

(4) $\phi(t) - M(|x| + \|u\|) \le L(t,x,u)$ a.e. for some $\phi(\cdot) \in L^1$, $M \ge 0$.

THEOREM 2.2. *If the hypotheses $H(A)$, $H(E)$, $H(B)$, $H(U)$, $H(L)_2$ and H_a hold, then the problem $(*)_2$ admits an optimal "state-control" pair.*

Proof. Let $\{(x_n,u_n)\}_{n\ge 1} \subseteq L^2(X) \times L^2(Y)$ be a minimizing sequence of admissible "state-control" pairs. From the proof of Theorem 2.1, we know that, by passing to a subsequence if necessary, we may assume that $x_n \xrightarrow{w} x$ in $L^2(X)$, $u_n \xrightarrow{w} u$ in $L^2(Y)$, (x,u) is admissible too and for every $t \in T$, $E^{1/2}x_n(t) \xrightarrow{s} E^{1/2}x(t)$ in H. So $Ex_n(t) \xrightarrow{s} Ex(t)$ in H for all $t \in T$. Invoking Theorem 2.1 of Balder [**2**], we get

$$\begin{aligned}
&J_2(x,u) \le \underline{\lim}\, J_2(x_n,u_n) = m_2\\
&\Longrightarrow J_2(x,u) = m_2\\
&\Longrightarrow (x,u) \ \text{ is the desired optimal pair for } (*)_2 \quad \text{Q.E.D.}
\end{aligned}$$

3. Sensitivity analysis. In this section we present a sensitivity (stability) result of the optimal value, as the data of the problem change. Consider the following sequence of optimal control problems:

$$\begin{aligned}
&J_n(x_n,u_n) = \int_0^b L_n(t, Ex_n(t), u_n(t))\,dt \to \inf = m_n\\
&\text{s.t.}\quad dEx_n(t)/dt + A_n(t,x_n(t)) = (\hat{B}_n u_n)(t) \quad \text{a.e.}\\
&\qquad x_n(0) = x_0^n \in H\\
&\qquad u_n(t) \in U_n(t) \quad \text{a.e.;} \quad u_n(\cdot) \text{ is measurable}
\end{aligned} \tag{$(**)_n$}$$

and the limit problem

$$\begin{aligned}
&J(x,u) = \int_0^b L(t, Ex(t), u(t))\,dt \to \inf = m\\
&\text{s.t.}\quad dEx(t)/dt + A(t,x(t)) = (\hat{B}u)(t) \quad \text{a.e.}\\
&\qquad x(0) = x_0^n \in H\\
&\qquad u(t) \in U(t) \quad \text{a.e.;} \quad u(\cdot) \text{ is measurable}
\end{aligned} \tag{$(**)$}$$

We will need the following hypotheses on the data of the above problems:

$H(A)_1$: $A_n, A : T \times X \to X^*$ are operators s.t.

(1) $t \to A_n(t,x), A(t,x)$ are measurable,

(2) $x \to A_n(t,x), A(t,x)$ are hemicontinuous, strictly monotone,

(3) $\|A_n(t,x)\|_*, \|A(t,x)\|_* \le a(t) + b\|x\|$ a.e. $a(\cdot) \in: L^2_+$, $b > 0$,

(4) $\langle A_n(t,x), x\rangle, \langle A(t,x), x\rangle \ge c\|x\|^2$ a.e., $c > 0$,

(5) for every $x(\cdot) \in L^2(X)$ s.t. $E^{1/2}x(\cdot) \in C(T,H)$ and $d(Ex(\cdot))/dt \in L^2(X^*)$ we have $A_n(\cdot, x(\cdot)) \xrightarrow{s} A(\cdot, x(\cdot))$ in $L^2(X^*)$.

$H(B)_1$: $B_n, B : L^2(Y) \to L^2(H)$ are completely continuous and $B_n \xrightarrow{o} B$ (o denoting the operator norm topology).

$H(U)_1$: $U_n, U : T \to P_{\mathrm{fc}}(Y)$ are measurable multifunctions, L^2-integrably bounded by $\psi(\cdot) \in L^2_+$ (i.e. for all $n \ge 1$, $|U_n(t)| \le \psi(t)$ a.e.) and $U_n(t) \xrightarrow{\text{K-M}} U(t)$ a.e.

$H(L)_3$: $L_n, L : T \times H \times Y \to \mathbb{R}$ are integrands s.t. for all $n \ge 1$

(1) $t \to L_n(t,x,u)$ measurable

(2) $(x,u) \to L_n(t,x,u)$ is convex,

(3) $\phi_1(t) + M_1(|x|^2 + \|u\|^2) \le L_n(t,x,u) \le \phi_2(t) + M_2(|x|^2 + \|u\|^2)$ a.e. with $\phi_1, \phi_2 \in L^2$, $M_1, M_2 \in \mathbb{R}_+$.

(4) $L_n(t,\cdot,\cdot) \xrightarrow{\tau} L(t,\cdot,\cdot)$ a.e.

THEOREM 3.1. *If the hypotheses $H(A)_1$, $H(B)_1$, $H(U)_1$, $H(L)_3$ hold and $x_0^n \xrightarrow{s} x_0$ in H, then $m_n \to m$ as $n \to \infty$.*

Proof. Let (x,u) be an optimal admissible pair for the limit problem $(**)$. Its existence is guaranteed by Theorem 2.2. From the hypothesis $H(L)_3$ and Theorem 3.1 of Salvadori [**9**], we know that $J_n \xrightarrow{\tau} J$. Then from the definition of τ-convergence (Mosco [**5**]), we know that we can find $(y_n, v_n) \in L^2(H) \times L^2(Y)$ s.t. $(y_n, v_n) \xrightarrow{s \times s} (x,u)$ in $L^2(H) \times L^2(Y)$ s.t.

$$\lim J_n(y_n, v_n) = J(x,u).$$

Also from Theorem 4.4 of [**6**], we know that $S^2_{U_n} \xrightarrow{\text{K-M}} S^2_U$. Hence Theorem 3.33 of Attouch [**1**], tells us that $u_n = \mathrm{proj}(v_n; S^2_{U_n}) \xrightarrow{s} u$ in $L^2(Y)$. Let $x_n(\cdot) \in L^2(X)$ be the unique trajectory of $(**)_n$ generated by the admissible control $u_n(\cdot) \in L^2(Y)$ (uniqueness follows from the strict monotonicity of $A_n(t,\cdot)$). Then we have:

$$\begin{aligned}
&\langle d(Ex_n(t) - Ex(t))/dt, x_n(t) - x(t)\rangle \\
&\quad + \langle A_n(t, x_n(t)) - A(t, x(t)), x_n(t) - x(t)\rangle \\
&\quad = ((B_n u_n)(t) - (Bu)(t), x_n(t) - x(t)) \quad \text{a.e.} \\
\Longrightarrow\quad &\langle dE(x_n(t) - x(t))/dt x_n(t) - x(t)\rangle \\
&\quad + \langle A_n(t, x_n(t)) - A_n(t, x(t)), x_n(t) - x(t)\rangle \\
&\quad + \langle A_n(t, x(t)) - A(t, x(t)), x_n(t) - x(t)\rangle
\end{aligned}$$

$$
\begin{aligned}
&= ((B_n u_n)(t) - (Bu)(t), x_n(t) - x(t)) \quad \text{a.e.} \\
\Longrightarrow \quad & \langle dE(x_n(t) - x(t))/dt, x_n(t) - x(t)\rangle \\
&+ \langle A_n(t, x(t)) - A(t, x(t)), x_n(t) - x(t)\rangle \\
&\leq ((B_n u_n)(t) - (Bu)(t), x_n(t) - x(t)) \quad \text{a.e.}
\end{aligned}
$$

Integrating over $[0, t]$ and performing integration by parts, we get,

$$
\begin{aligned}
|E^{1/2} x_n(t) - E^{1/2} x(t)| &\leq |E^{1/2} x_0^n - E^{1/2} x_0| \\
&+ 2\int_0^t \|A_n(s, x(s)) - A(s, x(s))\|_* \cdot \|x_n(s) - x(s)\| \, ds \\
&+ 2\int_0^t |(B_n u_n)(s) - (Bu)(s)| \cdot \|x_n(s) - x(s)\| \, ds.
\end{aligned}
$$

Applying the Cauchy-Schwartz inequality on the integrals of the right-hand side, we get

$$
\begin{aligned}
|E^{1/2} x_n(t) - E^{1/2} x(t)| &\leq |E^{1/2} x_0^n - E^{1/2} x_0| \\
&+ 2\|\hat{A}_n(x) - \hat{A}(x)\|_{L^2(X^*)} \cdot 2M_1' + 2\|B_n u_n - Bu\|_{L^2(H)} \cdot 2M_1'
\end{aligned}
$$

(recall that from the proof of Theorem 2.1 — in particular inequality (3) — we have $\|x_n\|_{L^2(X)}, \|x\|_{L^2(X)} \leq M_1'$ for all $n \geq 1$).

From hypothesis $H(A)_1(5)$ we have $2\|\hat{A}_n(x) - \hat{A}(x)\|_{L^2(X^*)} \to 0$ as $n \to \infty$. Also note that (see hypothesis $H(B)_1$)

$$
\begin{aligned}
\|B_n u_n - Bu\|_{L^2(H)} &\leq \|B_n u_n - Bu_n\|_{L^2(H)} + \|Bu_n - Bu\|_{L^2(H)} \\
&\leq \|B_n - B\|_{\mathcal{L}} \|\psi\|_2 + \|Bu_n - Bu\|_{L^2(H)} \to 0.
\end{aligned}
$$

So $E^{1/2} x_n(\cdot) \to E^{1/2} x(\cdot)$ in $C(T, H)$ implies that $Ex_n(\cdot) \to Ex(\cdot)$ in $C(T, H)$. Furthermore $\{J_n\}_{n\geq 1}$ is a sequence of convex integral functionals, uniformly bounded in every ball in $L^2(H) \times L^2(Y)$. So, from a well known result of convex analysis (see for example Rockafellar [**8**]), we know that $\{J_n\}_{n\geq 1}$ is locally equi-Lipschitzian. Thus we have:

$$
\begin{aligned}
&|J_n(y_n, v_n) - J_n(x_n, u_n)| \leq k[\, \|Ey_n - Ex_n\|_{L^2(H)} + \|v_n - u_n\|_{L^2(Y)}] \to 0, \quad k > 0 \\
&\Longrightarrow \quad \lim J_n(x_n, u_n) = J(x, u) = m \\
&\Longrightarrow \quad \overline{\lim}\, m_n \leq m. \qquad (1)
\end{aligned}
$$

On the other hand let $\{(x_n, u_n)\}_{n\geq 1}$ be a sequence of optimal pairs for the approximating problems $(**)_n$. By passing to a subsequence if necessary, we may assume that $u_n \xrightarrow{w} u$ in $L^2(Y)$. Since $S^2_{U_n} \xrightarrow{\text{K-M}} S^2_U$, we will have that $u \in S^2_U$. Let $x(\cdot)$ be the unique trajectory of the limit problem $(**)$, generated by the admissible control $u(\cdot)$. As before, we can show that $Ex_n \to Ex$ in $C(T, H)$. Since $J_n \xrightarrow{\tau} J$, from the definition of the τ-convergence, we have

$$
J(x, u) \leq \underline{\lim}\, J_n(x_n, u_n) \quad \Longrightarrow \quad m \leq \underline{\lim}\, m_n. \qquad (2)
$$

From (1) and (2) above, we conclude that $m_n \to m$. Q.E.D.

4. An example. In this section, we present an example of a degenerate parabolic optimal control problem, illustrating the applicability of our work. So, let Z be a bounded domain in $\mathbb{R}^n$ with smooth boundary $\Gamma = \partial Z$. The optimal control problem under consideration, is the following:

$$\begin{aligned}
&J(x,u) = \int_0^b \int_Z L(t,z,\theta(z)x(t,z),u(t,z))\,dz dt \to \inf = m \\
&\text{s.t.}\quad \theta(z)\frac{\partial x(t,z)}{\partial t} + \sum_{|a|\le m} (-1)^{|a|} D^a A_a(t,z,\eta(x(t,z))) \\
&\qquad = \int_0^b \int_Z k(t,s,z,z')u(s,z')\,dz'ds; \\
&\qquad D^\beta x|_{T\times\Gamma} = 0,\quad |\beta| \le m-1,\quad x(0,z) = x_0(z) \\
&\qquad \int_Z u(t,z)^2\,dz \le r(t)^2;\quad u(\,\cdot\,,\,\cdot\,) \text{ — measurable.}
\end{aligned} \tag{$***$}$$

Here $\alpha = (\alpha_1, \ldots, \alpha_n)$ is a multi-index, $|\alpha| = \sum_{k=1}^n \alpha_k$ is the length of the multi-index and $\eta(x(z)) = \{D^\alpha x(z) : |\alpha| \le m\}$. We will need the following hypotheses concerning the data of $(***)$:

$H(A)_2$: $A_\alpha : T \times Z \times \mathbb{R}^{n_m} \to \mathbb{R}$ ($n_m = (n+m)!/(n!m!)$) are functions s.t.

(1) for every η, $(t,z) \to A_\alpha(t,z,\eta)$ is measurable,
(2) for every $(t,z) \in T \times Z$, $\eta \in A(t,z,\eta)$ is continuous,
(3) $|A_\alpha(t,z,\eta)| \le a(t,z) + b(z)\|\eta\|$ a.e. with $a(\,\cdot\,,\,\cdot\,) \in L^2(T\times Z)_+$, $b(\,\cdot\,) \in L^\infty(Z)_+$,
(4) $\sum_{|\alpha|\le m}(A_\alpha(t,z,\eta) - A_\alpha(t,z,\eta'))(\eta_\alpha - \eta'_\alpha) \ge 0$ for all $(t,z) \in T\times Z$ and $\eta, \eta' \in \mathbb{R}^{n_m}$,
(5) $\sum_{|\alpha|\le m} A_\alpha(t,z,\eta)\eta_\alpha \ge c\sum_{|\alpha|\le m}\eta_\alpha^2$ a.e., $c > 0$.

$H(k)$: $k \in L^2(T\times T\times Z\times Z)$,

$H(r)$: $r(\,\cdot\,) \in L^2(T)_+$,

$H(\theta)$: $\theta(\,\cdot\,) \in L^\infty(Z)$, $\theta \ge 0$,

$H(L)_4$: $L : T\times Z\times\mathbb{R}\times\mathbb{R} \to \overline{\mathbb{R}}$ is an integrand s.t.

(1) $L(\,\cdot\,,\,\cdot\,,\,\cdot\,,\,\cdot\,)$ is measurable,
(2) for every $(t,z) \in T\times Z$, $(x,u) \to L(t,z,x,u)$ is l.s.c and convex in u,
(3) $\phi(t,z) - M(z)(|x| + |u|) \le L(t,z,x,u)$ a.e. with $\phi \in L^1(T\times Z)$, $M \in L^\infty(Z)_+$.

Here $H = L^2(Z)$, $X = H_0^m(Z)$ and $X^* = H^{-m}(Z) = (H_0^m(Z))^*$. From the well known Sobolev embedding theorem, we know that (X, H, X^*) is a Gelfand triple with all embeddings being compact.

Let $a : T \times H_0^m(Z) \times H_0^m(Z)$ be the time dependent Dirichlet form, associated with the nonlinear, elliptic partial differential operator of our problem. So we have

$$a(t,x,y) = \sum_{|\alpha|\le m} \int_Z A_\alpha(t,z,\eta(x(z)))D^\alpha y(z)\,dz$$

Using Cauchy's and Minkowski's inequalities, we get

$$\begin{aligned}
&\int_Z A_\alpha(t,z,\eta(x(z)))D^\alpha y(z)\,dz \\
&\le \left(\int_Z |A_\alpha(t,z,\eta(x(z)))|^2\,dz\right)^{1/2}\left(\int_Z |D^\alpha y(z)|^2\,dz\right)^{1/2} \\
&\le \left[\left(\int_Z a(t,z)^2\,dz\right)^{1/2} + \|b\|_\infty \sum_{|\gamma|\le m}\left(\int_Z D^\gamma x(z)^2\,dz\right)^{1/2}\right]\left(\int_Z |D^\alpha y(z)|^2\,dz\right)^{1/2}.
\end{aligned}$$

Summing over $|\alpha| \le m$, we get $|a(t,x,y)| \le (\hat{a}(t) + \hat{b}(t)\|x\|_{H_0^m(Z)})\cdot\|y\|_{H_0^m(Z)}$ where $\hat{a}(\cdot) = \|a(t,\cdot)\|_2 \in L^2(T)_+$ and $\hat{b} = \|b\|_\infty$. So there exists a generally nonlinear operator $\hat{A}(t,\cdot) : X \to X^*$ s.t. $a(t,x,y) = \langle A(t,x), y\rangle$, where $\langle\cdot,\cdot\rangle$ denotes the duality brackets for $(H_0^m(Z), H^{-m}(Z))$.

Clearly from the Fubini's theorem, a $a(\cdot,x,y)$ is measurable. So $A(\cdot,x)$ is weakly measurable and since $H^{-m}(Z)$ is separable, from the Pettis measurability theorem, we deduce that $A(\cdot,x)$ is measurable. Also if $x_n \xrightarrow{s} x$ in $H_0^m(Z)$, then

$$\begin{aligned}
&\langle \hat{A}(t,x_n) - \hat{A}(t,x), y\rangle \\
&= \sum_{|\alpha|\le m}\int_Z |A_\alpha(t,z,\eta(x_n(z))) - A_\alpha(t,z,\eta(x(z)))|\cdot|D^\alpha y(z)|\,dz \to 0 \\
&\implies \quad \hat{A}(t,\cdot) \text{ is demicontinuous, hence hemicontinuous.}
\end{aligned}$$

Furthermore from the hypothesis $H(A)_2(4)$, we have $\langle \hat{A}(t,x) - \hat{A}(t,y), x-y\rangle \ge 0$, while from the hypothesis $H(A)_2(5)$, we have $\langle \hat{A}(t,x), x\rangle \ge \hat{c}\|x\|^2_{H_0^m(Z)}$, $\hat{c} > 0$. Thus we have checked that the operator $\hat{A}(t,x)$ satisfies the hypothesis $H(A)$.

Next let $B : L^2(T, L^2(Z)) \to L^2(T, L^2(Z))$ be defined by

$$(Bu)(t,z) = \int_0^b \int_Z k(t,s,z,z')u(s,z')\,dz'ds.$$

From the Krasnoselski-Ladyzenskaya theorem (see Martin [**4**]), we know that $B(\cdot)$ is a completely continuous operator on $L^2(T, L^2(Z)) = L^2(T \times Z)$. Also let $E \in \mathcal{L}(L^2(Z))_+$ be defined by $(Eu)(z) = \theta(z)u(z)$ and set $U(t) = \{u \in L^2(Z) = Y : \|u\|_2 \le r(t)\}$. Clearly $U(\cdot)$ is measurable and $|U(t)| \le r(t)$ a.e. For the cost functional we set for $(x,u) \in L^2(Z) \times L^2(Z)$:

$$\hat{L}(t, Ex, u) = \int_Z L(t,z,\theta(z)x(z),u(z))\,dz.$$

Let L_k be Caratheodory integrands s.t. $L_k \uparrow L$ and $\phi(t,z) - M(z)(|x| + |u|) \le L_k(t,z,x,u) \le k$ (see for example Pappas [**7**]). Then let $\hat{L}_k(t,Ex,u) = \int_Z L(t,z,\theta(z)x(z),u(z))\,dz$. Clearly $\hat{L}_k(t,y,u)$ is a Caratheodory function (i.e. measurable in t, continuous in (y,u)) and so is jointly measurable. Also from the monotone convergence theorem, we have that $\hat{L}_k \uparrow \hat{L}$ and hence $\hat{L}$ is jointly measurable. Furthermore $\hat{L}(t,\cdot,\cdot)$ is l.s.c. Function $\hat{L}(t,x,\cdot)$ is convex and $\hat{\phi}(t) - \hat{M}(|y| + \|u\|) \le \hat{L}(t,y,u)$ a.e. with $\hat{\phi}(t) = \|\phi(t,\cdot)\|_2$ and $\hat{M} = \|M\|_\infty$. So we have satisfied the hypothesis $H(L)_2$.

Rewrite the problem $(***)$, in the following equivalent abstract form:

$$\begin{aligned} &\hat{J}(x,u) = \int_0^b \hat{L}(t,Ex(t),u(t))\,dt \to \inf = \hat{m} \\ \text{s.t.}\quad & d(Ex(t))/dt + \hat{A}(t,x(t)) = (Bu)(t) \quad \text{a.e.} \\ & x(0) = x_0(\cdot) \\ & u(t) \in U(t) \text{ a.e.}; \quad u(\cdot) \text{ is measurable} \end{aligned} \qquad (***)'$$

All the hypotheses of Theorem 2.2 have been verified. So, applying Theorem 2.2, we get:

THEOREM 4.1. *If the hypotheses $H(A)_2$, $H(k)$, $H(r)$, $H(\theta)$, $H(L)_4$ hold and $x_0(\cdot) \in L^2(Z)$, then $(***)$ admits an optimal pair $(x,u) \in L^2(T,H_0^m(Z)) \times L^2(T \times Z)$ and $\sqrt{\theta(z)}\,x(t,z)$ belongs to $C(T,L^2(Z))$.*

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