PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série tome 51 (65), 1992, 115–120

ON THE CLASSIFICATION OF TOTALLY UMBILICAL CR-SUBMANIFOLDS OF A KAEHLER MANIFOLD

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Abstract. We show that three-dimensional totally umbilical proper CR-submanifolds of a Kaehler manifold are extrinsic spheres. Thus we extend a classification theorem of these submanifolds for dimension less than five.

1. Introduction. The notion of CR-submanifold of a Kaehler manifold was introduced by Bejancu [**1**]. Let $\overline{M}$ be an m-dimensional Kaehler manifold with almost complex structure J. A $(2p+q)$-dimensional submanifold M of $\overline{M}$ is called a CR-submanifold if there exists a pair of orthogonal complementary distributions D and $D^{\perp}$ such that $JD = D$ and $JD^{\perp} \subset \nu$, where ν is the normal bundle of M and $\dim D = 2p$, $\dim D^{\perp} = q$. Thus the normal bundle ν splits as $\nu = JD^{\perp} \oplus \mu$, where μ is an invariant sub-bundle of ν under J. A CR-submanifold is said to be proper if neither $D = \{0\}$ nor $D^{\perp} = \{0\}$.

Bejancu considered totally umbilical CR-submanifolds of a Kaehler manifold. he proved that if $\dim D^{\perp} > 1$, then these submanifolds ar totally geodesic [**2**]. Blair and Chen [**3**] and later Deshmukh and Husain have also studied these submanifolds. In fact Deshmukh and Husain have proved a classification theorem for totally umbilical CR-submanifolds provided that $\dim M \geq 5$. Their theorem is the following [**4**].

THEOREM 1. *Let M, $(\dim M \geq 5)$ be a complete simply connected totally umbilical CR-submanifold of a Kaehler manifold $\overline{M}$. Then M is one of the following*:

(i) *locally the Riemannian product of a holomorphic and a totally real submanifold of $\overline{M}$*;

(ii) *totally real submanifold*;

(iii) *isometric to an ordinary sphere*;

(iv) *homothetic to a Sasakian manifold.*

AMS Subject Classification (1985): 53 C 40, secondary 53 C 55

In this paper we consider the case where $\dim M = 3$. For this case we obtain the following theorem:

THEOREM 2. *Let M be a 3-dimensional totally umbilical proper CR-submanifold of a Kaehler manifold $\overline{M}$. Then M is an extrinsic sphere.*

Note that, since M in the above theorem is proper and 3-dimensional, then $\dim D^{\perp} = 1$. However if $\dim M = 4$ and M is proper, then $\dim D^{\perp} = 2$, and in this case one may use the result in [**2**] to conclude that M is totally geodesic. Using this and a result in [**3**] we conclude that if $\dim M = 4$, then M is locally a Riemannian product of a holomorphic submanifold and a totally real submanifold of $\overline{M}$. If $\dim M = 3$, then Theorem 2 and a result in [**5**] imply that M is either (iii) or (iv) of Theorem 1. Note that for $\dim M = 2$ or 1, M is either a holomorphic submanifold or a totally real submanifold. Thus a complete classification of totally umbilical CR-submanifolds of a Kaehler manifold is obtained.

2. Preliminaries. We shall denote by $\overline{\nabla}$, ∇, $\nabla^{\perp}$ the Riemannian connection on $\overline{M}$, M and the normal bundle respectively. They are related as follows:

$$\overline{\nabla}_X Y = \nabla_X Y + h(X,Y) \tag{2.1}$$

$$\overline{\nabla}_X N = -A_N X + \nabla^{\perp}_X N, \qquad N \in \nu. \tag{2.2}$$

where $h(X,Y)$ and $A_N X$ are the second fundamental forms which are related by

$$g(h(X,Y),N) = g(A_N X, Y) \tag{2.3}$$

where X and Y are vector fields on M.

Now let $\overline{R}$, R and $R^{\perp}$ be the curvature tensors associated with $\overline{\nabla}$, ∇ and $\nabla^{\perp}$ respectively. The curvature tensor $\overline{R}$ satisfies

$$\overline{R}(JX,JY)Z = \overline{R}(X,Y)Z, \qquad \overline{R}(X,Y)JZ = J\overline{R}(X,Y)Z. \tag{2.4}$$

If X, Y, Z, W are vector fields on M, then Gauss and Codazzi equations are respectively given by

$$\begin{aligned} R(X,Y;Z,W) = \overline{R}(X,Y;Z,W) &+ g(h(X,W),h(Y,Z)) \\ &- g(h(X,Z),h(Y,W)) \end{aligned} \tag{2.5}$$

$$\overline{R}(X,Y;Z,N) = g((\overline{\nabla}_X h)(Y,Z) - (\overline{\nabla}_Y h)(X,Z)N), \tag{2.6}$$

where

$$\overline{R}(X,Y;Z,N) = g(\overline{R}(X,Y)Z,N)$$
$$(\overline{\nabla}_X h)(Y,Z) = \nabla^{\perp}_X h(Y,Z) - h(\nabla_X Y, Z) - h(\nabla_X Z, Y).$$

A CR-submanifold is said to be totally umbilical if $h(X,Y) = g(X,Y)H$, where $H = (\operatorname{trace} h)/n$ is the mean curvature vector.

A totally umbilical submanifold of a Riemannian manifold which has nonzero parallel mean curvature vector (i.e. $\nabla^{\perp}_X H = 0$) is called an extrinsic sphere. If M is totally umbilical CR-submanifold, the equations (2.1), (2.2) and (2.6) become

$$\overline{\nabla}_X Y = \nabla_X Y + g(X,Y)H \tag{2.7}$$

$$\overline{\nabla}_X N = -g(H,N)X + \nabla^{\perp}_X N \tag{2.8}$$

$$\overline{R}(X,Y;Z,N) = g(Y,Z)g(\nabla^{\perp}_X H, N) - g(X,Z)g(\nabla^{\perp}_Y H, N). \tag{2.9}$$

Bianchi's first and second identities are given respectively by

$$R(X,Y,Z) + R(Y,Z,X) + R(Z,X,Y) = 0 \tag{2.10}$$

$$(\nabla_X R)(Y,Z) + (\nabla_Y R)(Z,X) + (\nabla_Z R)(X,Y) = 0. \tag{2.11}$$

3. Three-dimensional totally umbilical CR-submanifold of Kaehler manifold. We consider a 3-dimensional totally umbilical proper CR-submanifold M of a Kaehler manifold $\overline{M}$. Then we prove the following lemmas

LEMMA 1. *$H \in JD^{\perp}$, and for $z \in D^{\perp}$, $\nabla^{\perp}_X H = 0$.*

Proof. Since M is proper and 3-dimensional, $\dim D = 2$ and $\dim D^{\perp} = 1$. For X, Y in D the equation $J\overline{\nabla}_X Y = \overline{\nabla}_X JY$ and (2.7) give

$$J\nabla_X Y + g(X,Y)JH = \nabla_X JY + g(X,JY)H.$$

Taking inner product with $N \in \mu$, we get

$$g(X,Y)g(JH,N) = g(X,JY)g(H,N)$$

With $Y = JX$ in the above equation, we have

$$\|X\| g(H,N) = 0, \qquad \text{i.e.} \qquad H \in JD^{\perp}.$$

To prove the second part of the Lemma, let $N \in \mu$. Then it follows from (2.9) that $\overline{R}(Z,X;JX,JN) = 0$ for $X \in D$. Using (2.4) in this equation we get $\overline{R}(Z,X;X,N) = 0$. Using (2.9) in this last equation we have $g(\nabla^{\perp}_Z H, N) = 0$, from which it follows that $\nabla^{\perp}_Z H \in JD^{\perp}$. We need to show that $\nabla^{\perp}_Z H \in \mu$. From (2.9) and (2.4) we get $\overline{R}(Z,X;X,Z) = \overline{R}(Z,X;JX,JZ) = 0$. Using linearity of $\overline{R}$, we then get $\overline{R}(Z,X;JX,Z) = 0$. From this it follows that $\overline{R}(Z,X,X,JZ) = 0$. Now using (2.9) the last equation gives $g(\nabla^{\perp}_Z H, JZ) = 0$, i.e. $\nabla^{\perp}_Z H \in \mu$. Thus $\nabla^{\perp}_Z H \in JD \cap \mu = \{0\}$. This finishes the proof of Lemma 1.

LEMMA 2. *Let $\{X, JX, Z\}$ be an orthonormal frame field on M where $X \in D$ and $Z \in D^{\perp}$. Then we have the following equations*

$$\begin{array}{lll} \nabla_X X = aJX, & \nabla_{JX} X = -bJX + \alpha Z, & \nabla_Z X = cJX, \\ \nabla_X JX = -aX - \alpha Z, & \nabla_{JX} JX = bX, & \nabla_Z JX = -cX, \\ \nabla_X Z = \alpha JX, & \nabla_{JX} Z = -\alpha X, & \nabla_Z Z = 0, \end{array}$$

where a, b, c are smooth functions on M and $\alpha = \|H\|$.

Proof. We know from Lemma 1 that $H \in JD^{\perp}$. Since dim $JD^{\perp} = 1$, one can write $H = \alpha JZ$ for some smooth function α on M. Since M is totally umbilical we get

$$h(X,X) = h(JX,JX) = h(Z,Z) = \alpha JZ \tag{3.1}$$

$$A_{JZ}X = \alpha X, \quad A_{JZ}JX = \alpha JX, \quad A_{JZ}Z = \alpha Z, \tag{3.2}$$
$$h(X,JX) = h(X,Z) = h(Z,JX) = 0.$$

Using the equation (2.7) and (2.8) in the equation $\overline{\nabla}_Z JZ = J\overline{\nabla}_Z Z$ and taking inner product with $W \in D$, we get $g(\nabla_Z, W) = 0$, i.e. $\nabla_Z Z \in D^{\perp}$. Since $g(Z,Z) = 1$ we also have $\nabla_Z Z \in D$. Therefore we have

$$\nabla_Z Z = 0. \tag{3.3}$$

Using (3.3) we have

$$g(\nabla_Z X, Z) = 0, \qquad g(\nabla_Z JX, Z) = 0. \tag{3.4}$$

Also using the equation $(\overline{\nabla}_X J)(Z) = 0$ and (3.2) we get

$$g(\nabla_X Z, X) = 0, \qquad g(\nabla_X Z, JX) = \alpha. \tag{3.5}$$

Now using the equation $(\overline{\nabla}_{JX} J)(Z) = 0$ we have

$$g(\nabla_{JX} Z, X) = -\alpha, \qquad g(\nabla_{JX} Z, JX) = 0. \tag{3.6}$$

Similarly the equations $(\overline{\nabla}_X J)(X) = 0$, $(\overline{\nabla}_{JX} J)(X) = 0$ with the help of (3.1) give

$$g(\nabla_X X, Z) = 0, \qquad g(\nabla_{JX} JX, Z) = 0. \tag{3.7}$$

The lemma follows from the equations, (3.3), (3.4), (3,5), (3.6) and (3.7).

LEMMA 3. *Let $\{X, JX, Z\}$ be the orthonormal frame field on M. Then we have the following expressions for the curvature tensor of M*

$$\begin{aligned}
R(X,Z,Z) &= \alpha^2 X, \quad R(JX,Z,Z) = \alpha^2 JX\\
R(JX,Z,X) &= (JX(c) + \alpha a - ca + Z(b))JX\\
R(X,JX,Z) &= -X(\alpha)X - JX(\alpha)JX\\
R(Z,X,JX) &= (X(c) - Z(a) + \alpha b - cb)X\\
R(Z,X,X) &= -(X(c) + \alpha b - Z(a) - cb)JX + \alpha^2 Z\\
R(Z,JX,JX) &= (JX(c) + Z(b) + \alpha a - ca)X + \alpha^2 Z
\end{aligned}$$

Proof. Using Lemma 2 and the definition of the curvature tensor R, $R(X,Y,Z) = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z$, we get Lemma 3.

Proof of Theorem 2. Using the expressions for the curvature tensor given by Lemma (3) in Bianchi first identity (2.10) we get

$$[X(c) - Z(a) + \alpha b - cb - X(\alpha)]X + [JX(c) + \alpha a - ca + Z(b) - JX(\alpha)]JX = 0$$

from which it follows that

$$X(c) - Z(a) + \alpha b - cb - X(\alpha) = 0 \tag{3.8}$$

$$JX(c) + \alpha a - ca + Z(b) - JX(\alpha) = 0 \tag{3.9}$$

Applying Bianchi's second identity (2.11) to Z we have

$$(\nabla_X R)(JX, Z)Z + (\nabla_{JX} R)(Z, X)Z + (\nabla_Z R)(X, JX)Z = 0 \tag{3.10}$$

where

$$\begin{aligned}(\nabla_X R)(JX, Z)Z = \nabla_X R(JX, Z)Z - R(\nabla_X JX, Z)Z \\ - R(JX, \nabla_X Z)Z - R(JX, Z)\nabla_X Z.\end{aligned}$$

Using Lemma 3 and Lemma 2 we obtain

$$\begin{aligned}(\nabla_X R)(JX, Z)Z &= 2\alpha X(\alpha)JX + \alpha[JX(c) + Z(b) + \alpha a - ca]X \\ &= 2\alpha X(\alpha)JX + \alpha JX(\alpha)X,\end{aligned} \tag{3.11}$$

where we have used (3.9) to get the last equality. Similarly we get

$$(\nabla_{JX} R)(Z, X)Z = -2\alpha JX(\alpha)X \tag{3.12}$$

$$(\nabla_Z R)(X, JX)Z = (cJX(\alpha) - ZX(\alpha))X - (cX(\alpha) + ZJX(\alpha))JX. \tag{3.13}$$

Now using (3.11), (3.12) and (3.13) in (3.10) we found that the X-components and the JX components give respectively

$$(c - \alpha)JX(\alpha) - ZX(\alpha) = 0 \tag{3.14}$$

$$(2\alpha - c)X(\alpha) - ZJX(\alpha) = 0. \tag{3.15}$$

Using the equation $\overline{\nabla}_V JZ = J\overline{\nabla}_V Z$, where $V = X$, JX or Z with the help of (3.2) and Lemma 2 we get $\nabla^{\perp}_V JZ = 0$. Since $\nabla^{\perp}_Z H = 0$, from Lemma 1, and $H = \alpha JZ$ we have

$$Z(\alpha) = 0. \tag{3.16}$$

Therefore, the equation $[Z, JX](\alpha) = [\nabla_Z JX - \nabla_{JX} Z](\alpha)$ implies that $ZJX(\alpha) = (\nabla_Z JX - \nabla_{JX} Z)(\alpha)$. Using Lemma 2 in this equation we get

$$ZJX(\alpha) = (\alpha - c)X(\alpha). \tag{3.17}$$

Using (3.17) in (3.15) we have

$$\alpha X(\alpha) = 0. \tag{3.18}$$

Now if we repeat the above arguments for the orthonormal frame field $\{W, JW, Z\}$, where $W = -JX$, we get the result in (3.18) for W with the same α as M is totally umbilical i.e. we get $\alpha W(\alpha) = 0$, or

$$\alpha JX(\alpha) = 0. \tag{3.19}$$

Equations (3.16), (3.18) and (3.19) imply that α^2 is constant. i.e. α is constant. Using this and $\nabla^{\perp}_V JZ = 0$ for $V = X$, JX or Z we get $\nabla^{\perp}_V H = 0$ i.e. M is an extrinsic sphere.

Now we have the following theorem.

THEOREM 3 [**2**]. *Let M be a totally umbilical 4-dimensional proper CR-submanifold of a Kaehler manifold $\overline{M}$. Then M is totally geodesic.*

COROLLARY. *Let M be as in Theorem 2 or Theorem 3. If* $\dim M = 4$*, then M is locally the Riemannian product of a holomorphic submanifold and a totally real submanifold of $\overline{M}$. If* $\dim M = 3$ *then M is either* (iii) *or* (iv) *of Theorem* 1.

Proof. The first part of the corollary follows from Theorem 3 and a result of [**3**]. The second part follows from Theorem 2 and a result of [**5**].

Thus Theorem 1 is extended for $\dim M < 5$.

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(Received 10 09 1991)