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SOME REMARKS ON ISOTROPIC SUBMANIFOLDS

Luc Vrancken*

Abstract. The notion of isotropic submanifolds of an arbitrary Riemannian manifold was first introduced by B. O'Neill. In this paper, we study n-dimensional, totally real, isotropic submanifolds of $\mathbb{C}P^n(4)$. These submanifolds have been previously studied by H. Naitoh, S. Montiel and F. Urbano under the additional assumption that M is complete. Here we prove some local classification theorems for totally real isotropic submanifolds of the complex projective space.

1. Introduction. The notion of isotropic submanifolds of an arbitrary Riemannian manifold was first introduced by B. O'Neill in [**O**]$_1$. In this paper, we study n-dimensional, totally real, isotropic submanifolds of $\mathbb{C}P^n(4)$. These submanifolds have been previously studied by Naitoh, Montiel, Urbano and Ejiri in [**N**], [**M–U**] and [**E**]. Naitoh proved the following result.

THEOREM A [**N**]. *Let M be an $n(\geq 2)$-dimensional complete totally real λ-isotropic submanifold in $\mathbb{C}P^n(4)$ with parallel second fundamental form. Then M is locally isometric with one of the following symmetric spaces*:

$$S^1 \times S^{n-1}, \quad \frac{\mathrm{SU}(3)}{\mathrm{SO}(3)}, \quad \mathrm{SU}(3), \quad \frac{\mathrm{SU}(6)}{\mathrm{Sp}(3)}, \quad \frac{E_6}{F_4}$$

Moreover $\lambda = 1/\sqrt{2}$.

Later, this result was generalized by S. Montiel, F. Urbano and N. Ejiri.

THEOREM B [**E**]. *Let M be a n-dimensional totally real isotropic submanifold of $\mathbb{C}P^n(4)$. If M is not totally geodesic, then M is one of the following*:

a minimal surface $(n = 2)$,

an Einstein minimal submanifold with parallel second fundamental form,

a conformally flat submanifold.

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*Research Assistant of the National Fund for Scientific Research (Belgium).

Theorem C [**M–U**]. *Let M^n be a complete, totally real, constant isotropic submanifold of $\mathbb{C}P^n(4)$. Then either M is totally geodesic or M is locally isometric with* $S^1 \times S^{n-1}$, $\dfrac{\mathrm{SU}(3)}{\mathrm{SO}(3)}$, $\mathrm{SU}(3)$, $\dfrac{\mathrm{SU}(6)}{\mathrm{Sp}(3)}$, $\dfrac{E_6}{F_4}$.

Notice that in Theorem A as well as in Theorem C the condition assumed that M is *complete*. Here, we generalize the results; indeed, we will show that the completeness condition in these theorems can be dropped. The main difference from the approach by H. Naitoh is that he used the theory of Lie groups and hence the completeness assumption has to be imposed while we will apply the local uniqueness theorem. In order to do so, we have to develop further the results of Montiel and Urbano.

Theorem 3.1. *Let M^n be a minimal, isotropic, totally real submanifold of $\mathbb{C}P^n(4)$. Then either M is totally geodesic or M is locally isometric with* $\dfrac{\mathrm{SU}(3)}{\mathrm{SO}(3)}$, $\mathrm{SU}(3)$, $\dfrac{\mathrm{SU}(6)}{\mathrm{Sp}(3)}$, $\dfrac{E_6}{F_4}$.

Theorem 3.2. *Let M be an n-dimensional totally real isotropic submanifold in $\mathbb{C}P^n(4)$. If M is not totally geodesic at the point p, then on a neighbourhood of p, M is locally isometric with one of the following manifolds*

a minimal surface $(n = 2)$,
$\dfrac{\mathrm{SU}(3)}{\mathrm{SO}(3)}$, $\mathrm{SU}(3)$, $\dfrac{\mathrm{SU}(6)}{\mathrm{Sp}(3)}$, E_6/F_4,
$S^1 \times S^{n-1}(3/2)$, $n > 2$,
$S^1 \times_d S^{n-1}$, *where d is defined in Section* 3, $n > 2$.

2. Preliminaries. In this section M will always denote an n-dimensional totally real submanifold of $\mathbb{C}P^n(4)$. We will denote the curvature tensor of M by R and the Ricci tensor by S. The formulas of Gauss and Weingarten are given by

$$D_X Y = \nabla_X Y + h(X,Y) \qquad \text{and} \qquad D_X \xi = -A_\xi X + \nabla^\perp_X \xi, \tag{2.1}$$

where X and Y are tangent vector fields and ξ is a normal vector field on M. The total reality condition then implies that

$$\nabla^\perp_X JY = J\nabla_X Y \qquad \text{and} \qquad A_{JX} Y = -Jh(X,Y). \tag{2.2}$$

The equations of Gauss, Codazzi and Ricci for a totally real submanifold of $\mathbb{C}P^n(4)$ are then given by

$$R(X,Y)Z = \langle Y,Z\rangle X - \langle X,Z\rangle Y + A_{h(Y,Z)}X - A_{h(X,Z)}Y, \tag{2.3}$$

$$(\nabla h)(X,Y,Z) = (\nabla h)(Y,X,Z), \tag{2.4}$$

$$\langle R^\perp(X,Y)\xi,\eta\rangle = \langle \widetilde{R}(X,Y)\xi,\eta\rangle + \langle [A_\xi, A_\eta]X, Y\rangle, \tag{2.5}$$

for tangent (resp. normal) vector fields X, Y and Z (resp. ξ and η) and $R^\perp$ (resp. $\widetilde{R}$) denotes the curvature tensor of $\nabla^\perp$ (resp. D). The total reality condition then implies for tangent vector fields X, Y, Z and W that

$$\langle R^\perp(X,Y)JZ, JW\rangle = \langle R(X,Y)Z, W\rangle, \tag{2.6}$$

$$\langle h(X,Y), JZ\rangle = \langle h(X,Z), JY\rangle. \tag{2.7}$$

From now on, we will also assume that M is an isotropic submanifold, i.e. in each point p of M, $\|h(v,v)\|$ is independent of the unit vector v. Hence, we obtain a function λ on M by

$$\lambda(p) = \|h(p,p)\|, \tag{2.8}$$

where $v \in UM_p$. In that case we obtain from $[\mathbf{O}]_1$ the following conditions for orthonormal tangent vectors X, Y, Z and W:

$$\langle h(X,Y), h(X,X)\rangle = 0, \tag{2.9}$$

$$\lambda^2 - \langle h(X,X), h(Y,Y)\rangle - 2\langle h(X,Y), h(X,Y)\rangle = 0, \tag{2.10}$$

$$\langle h(Y,Z), h(X,X)\rangle + 2\langle h(X,Y), h(X,Z)\rangle = 0, \tag{2.11}$$

$$\langle h(X,Y), h(Z,W)\rangle + \langle h(X,Z), h(W,Y)\rangle + \langle h(X,W), h(Y,Z)\rangle = 0. \tag{2.12}$$

3. Proof of the theorems. Let M be a n-dimensional, isotropic, totally real submanifold of $\mathbb{C}P^n$ and let $p \in M$. Then, from [**E**], we know that there are three cases to consider.

(i) p is a totally geodesic point (i.e. $h_p = 0$), or

(ii) There exists an orthonormal basis $\{e_1, e_2, \dots, e_n\}$ of T_pM such that

$$\begin{aligned} &\langle h(e_i,e_j), Je_k\rangle = 0, && \text{for } k \neq 1,\\ &\langle h(e_i,e_j), Je_1\rangle = -\delta_{ij}\lambda, && \text{for } i \neq 1,\\ &\langle h(e_1,e_1), Je_1\rangle = \lambda, \end{aligned}$$

where $\lambda > 0$, or

(iii) $n > 2$ and there exists an orthonormal basis $\{e_1, f_1, u_1, \dots, u_i, v_1, \dots, v_i, w_1, \dots, w_i\}$ such that $3i + 2 = n$ and

$$\begin{aligned} &h(e_1,e_1) = \lambda Je_1, && h(e_1,f_1) = -\lambda Jf_1, && h(f_1,f_1) = -\lambda Je_1,\\ &h(e_1,u_j) = -\lambda Ju_j, && h(e_1,v_j) = (\lambda/2)Jv_j, && h(e_1,w_j) = (\lambda/2)Jw_j,\\ &h(f_1,v_j) = (\sqrt{3}\lambda/2)Jv_j, && h(f_1,w_j) = -(\sqrt{3}\lambda/2)Jw_j, && h(u_j,u_i) = -\lambda Je_1, \end{aligned}$$
$$h(v_j,v_j) = (\lambda/2)Je_1 + (\sqrt{3}\lambda/2)\lambda Jf_1, \quad h(w_j,w_j) = (\lambda/2)Je_1 - (\sqrt{3}\lambda/2)Jf_1.$$

Furthermore, if $x, y, z \in \operatorname{vect}\{v_1, \dots, v_i, w_1, \dots, w_i\}$, then $\langle h(x,y), Jz\rangle = 0$; if $x, y \in \operatorname{vect}\{f_1, u_1, \dots, u_i\}$, then $h(x,y)$ has only a component in the direction of Je_1 and for every $x \in \operatorname{vect}\{v_1, \dots, v_i\}$ we have $\operatorname{vect}\{u_1, \dots, u_i\} = \{h(x,y) \mid y \in \operatorname{vect}\{w_1, \dots, w_i\}\}$. From these formulas, it is also immediately clear that M is minimal at the point p.

From [**E**] it also follows that if (iii) holds at a point p of M, then it holds on a neighbourhood of p and hence on the whole of M. Furthermore, we have from [**E**] that the Ricci curvature S is given by

$$S(v,w) = ((n-1) - (n+2)\lambda^2/2)\,\langle v,w\rangle.$$

Thus M is an Einstein space. Since $n > 2$, this then implies that λ is constant on M. Hence M is a minimal, totally real, constant isotropic submanifold of $\mathbb{C}P^n$. From Proposition 1 of [**M–U**] it then follows that M is parallel. On the other hand, by Theorem 1 of [**M–U**] we have that $n = 5, 8, 14$ or 26 (i.e. $m = 1, 2, 4$ or 8). Let us then denote $M_1 = \dfrac{\mathrm{SU}(3)}{\mathrm{SO}(3)}$, $M_2 = \mathrm{SU}(3)$, $M_3 = \dfrac{\mathrm{SU}(6)}{\mathrm{Sp}(3)}$ and $M_4 = E_6/F_4$. It is well-known, that there exists a parallel, minimal, totally real immersion of M_i into $\mathbb{C}P^{n(i)}$, where $n(i) = 3i+2$.

Lemma 3.1. *There exist constant λ^i_{jk}, $j,k,l \in \{1,2,\dots,n\}$ such that for every λ-isotropic, totally real immersion $f : M^n \to \mathbb{C}P^n$ which satisfies* (iii) *and for every point p of M, there exists an orthonormal basis $\{e_1,\dots,e_n\}$ of T_pM such that*

$$h(e_j,e_k) = \sum_{l=1}^{i} \lambda^l_{jk} J e_l$$

(*i.e. h is completely determined by λ*).

Proof. Let $\{e_1, f_1, u_1, \dots, u_i, v_1, \dots, v_i, w_1, \dots, w_i\}$ be an orthonormal basis which satisfies (iii). Then, we define $V_1 = \mathrm{vect}\{v_1,\dots,v_i\}$, $V_2 = \mathrm{vect}\{w_1,\dots,w_i\}$ and $V_3 = \mathrm{vect}\{u_1,\dots,u_i\}$. So, it is clear that these normed vector spaces are isomorphic. Let us denote the natural isomorphism between V_1 and V_2 by π_1 and the one between V_1 and V_3 by π_2. Then by (iii), we know that $h(v,\pi_1(v)) \in \pi_2(V_1)$. Furthermore from the isotropy conditions it follows for $v, w \in V_1$ that

$$\|h(v,\pi_1(w))\|^2 = (\sqrt{3}\lambda^2/2)\|v\|^2\,\|w\|^2.$$

Thus the mapping $A : V_1 \times V_1 \to V_1$ defined by

$$A(v,w) = -2/(\sqrt{3}\lambda)\pi_2(Jh(v,\pi_1(w))),$$

satisfies $\|A(v,w)\|^2 = \|v\|^2\|w\|^2$. Hence, by [**C**], A is either given by real multiplication or by complex multiplication or by quaternionic multiplication or by Cayley multiplication. Thus we can chose an orthonormal basis of V_1 such that, with respect to that basis, A is given by the traditional multiplication table for the real (resp. complex, quaternionic, Cayley) numbers. Combining this result with (iii) and (2.7) completes the proof of the lemma.

Lemma 3.2. *If* (iii) *holds at the point p, then $\lambda = \sqrt{2}/2$.*

Proof. We have seen that if (iii) holds at the point p, then it is satisfied in every point of M an M is a parallel submanifold of $\mathbb{C}P^n$. Since $(\nabla h) = 0$, we obtain that $R.h = 0$, where $(R.h)(X,Y,Z,W) = R^\perp(X,Y)h(Z,W) - h(R(X,Y,)Z,W) -$

$h(Z, R(X,Y)W)$. Then, if we take the basis given by (iii), we obtain in particular that

$$\begin{aligned} 0 &= \langle (R.h)(e_1, f_1, e_1, e_1), Jf_1\rangle \\ &= \langle R^{\perp}(e_1, f_1)JA_{Je_1}e_1, Jf_1\rangle - 2\langle h(R(e_1, f_1)e_1, e_1), Jf_1\rangle \\ &= -\lambda\langle R(e_1, f_1)f_1, e_1\rangle - 2\langle A_{Jf_1}e_1, R(e_1, f_1)e_1\rangle = -3\lambda(1-2\lambda^2). \end{aligned}$$

This completes the proof of this lemma.

This enables us to prove Theorem 3.1.

THEOREM 3.1. *Let M^n be a minimal, isotropic, totally real submanifold of $\mathbb{C}P^n$. Then either M is totally geodesic or $n = n(i)$ for some i and M is locally isometric with M_i.*

Proof. It is clear that in this case M is either totally geodesic or (iii) holds in every point p of M. Let us assume that M is not totally geodesic. Thus M is a parallel submanifold and $n = n(i)$ for some i. By Lemma 3.1, we know that in each point p there exists an orthonormal basis $\{e_1, \dots, e_n\}$ which satisfies all the conditions in Lemma 3.1.

Let us extend this basis by parallel translation along geodesics through p to a local orthonormal frame $\{E_1, \dots, E_n\}$ on a normal neighbourhood U of p. Then let $q \in U$ and let γ be the unique geodesic connecting p and q. Using the fact that $\nabla h = 0$ and the definition of the E_j, we obtain that

$$\gamma'\langle h(E_j, E_k), JE_l\rangle = \langle (\nabla h)(\gamma', E_j, E_k), JE_l\rangle = 0.$$

Therefore $\{E_1, \dots, E_n\}$ satisfies the conditions of Lemma 3.1 in every point g of U. Therefore, since $\lambda = 1/\sqrt{2}$, this determines the second fundamental form completely.

Since M_i can be parallel, minimal, totally real immersed in $\mathbb{C}P^{n(i)}(4)$ it is clear that for each point p' of M_i there exists a local orthonormal basis $\{F_1, \dots, F_{n(i)}\}$ defined by parallel translation along geodesics through p' such that all the conditions of Lemma 3.1 hold. Then, by numbering the basis vectors in an appropriate way, by Lemma 3.1 and the Gauss equation, we may assume that

$$\langle R(E_j, E_k)E_l, E_m\rangle = \langle R'(F_j, F_k)F_l, F_m\rangle,$$

where R' denotes the curvature tensor of M_i. Since all the vector fields are defined by parallel translation along geodesics through p (resp. p'), the Lemma of Cartan then implies that M is locally isometric with M_i.

Let us now assume that $p \in M$ is not a totally geodesic point. If (iii) holds in a point of M, then M is locally isometric with M_i. Thus, we may assume that (iii) does not hold in any point of M. Hence (ii) holds at p_0 and since p is not a totally geodesic point, (ii) must hold in a neighbourhood U of p. Let $q \in U$ and $\{e_1, \dots, e_n\}$ be an orthonormal basis of T_qM which satisfies (ii). Then, if $n > 2$ we can derive the sectional curvature of an arbitrary tangent plane π as follows. We may assume that π is spanned by $\cos\theta e_1 + \sin\theta u$ and by v, where e_1, u and v are orthonormal vectors. Then

$$K(\pi) = (1+\lambda^2) - 3\lambda^2\cos^2\theta.$$

Hence the sectional curvature varies between $(1+\lambda^2)$ and $(1-2\lambda^2)$ and is equal to $(1+\lambda^2)$ for every plane orthogonal to e_1. This argument shows that we can construct orthonormal vector fields $E_1, \dots, E_n$ on a neighbourhood U of p such that they satisfy (ii) in every point of U. It is not too difficult to see that this can be done also for $n = 2$, by looking at the number of maxima of $f(v) = \langle h(v,v), Jv\rangle$ at each point. Then from the Codazzi equations, see also [**E**], we obtain that

$$E_i(\lambda) = 0, \tag{3.1}$$

$$\nabla_{E_1} E_1 = 0, \tag{3.2}$$

$$\nabla_{E_i} E_1 = -E_1(\lambda)E_i/(3\lambda), \tag{3.3}$$

where $i > 1$. Thus we obtain two involutive distributions T_1 and T_2 given by

$$q \mapsto T_1(q) = \mathrm{vect}\{E_1\}, \qquad q \mapsto T_2(q) = \mathrm{vect}\{E_2, \dots, E_n\}.$$

Thus we can choose coordinates $(x_1, \dots, x_n)$ in a neighborhood of p such that $\partial/\partial x_1 = E_1$ and $\partial/\partial x_i \in \mathrm{vect}\{E_2, \dots, E_n\}$ and p corresponds to $(0, 0, \dots, 0)$. Then it follows from (3.1) that λ depends only on x_1. Furthermore, the hypersurfaces of M, given by $x_1 = c$, for c sufficiently small, have constant curvature $(1+\lambda(c)^2)$. Hence they are locally isometric with an $(n-1)$-dimensional sphere with radius $1/\sqrt{1+\lambda^2(c)}$. Then the metric of M can be written locally as

$$ds^2 = (dx_1)^2 + \sum_{i,j=2}^{n} g_{ij}(x_1, x_2, \dots, x_n)\, dx_i dx_j.$$

However, since all the hypersurfaces are isometric with a sphere, they are all homothetic. Thus $g_{ij}(x_1, \dots, x_n) = d(x_1)g_{ij}(0, x_2, \dots, x_n)$, where

$$d(x_1) = (1+\lambda(x_1)^2)/(1+\lambda(0)^2). \tag{3.4}$$

Hence M is locally isometric with the warped product $\mathbb{R} \times_d S^{n-1}(1+\lambda(0)^2)$ [**O**]$_2$. Furthermore, by applying the Gauss equation, (3.1), (3.2), (3.3) and (ii), we obtain that

$$\langle R(E_1, E_i)E_1, E_i\rangle = -(1-2\lambda(x_1)^2) \quad \text{and} \quad \langle R(E_1, E_i)E_1, E_i\rangle = -\frac{1}{3}\frac{\lambda''}{3\lambda} + \frac{4}{3}\frac{(\lambda')^2}{\lambda^2}.$$

Hence λ is a solution of the following differential equation: $3\lambda''\lambda - 4(\lambda')^2 = (1-2\lambda^2)\lambda^2$.

The solutions of this differential equation are then given by the following lemma.

Lemma 3.3. *The nonzero, positive solutions of the real differential equation* $3\lambda''\lambda - 4(\lambda')^2 = (1-2\lambda^2)\lambda^2$ *are given by* $\lambda(x_1) = 1/\sqrt{2}$, *and* $\lambda(x_1) = F^{-1}(x_1)$, *where* F *is defined by*

$$F(y) = -\frac{1}{3}\int_{\lambda_0}^{y} (Cx^{8/3} - x^2/9 - x^4/9)^{-1/2}\, dx$$

Proof. By putting $\lambda = f^{-3}$, we obtain that $f'' = -(1-2f^{-6})f/9$. So, if we write $f' = p(f)$ we obtain that $(p^2)' = -(2/9)(1-2f^{-6})f$. Thus $f' = p(f) = (C - (f^2+f^{-4})/9)^{1/2}$. Now, there are two possibilities. Either, $(C-(f^2+f^{-4})/9) \equiv 0$, in which case, we obtain that $\lambda = 1/\sqrt{2}$ or $\lambda = 0$, or $(C-(f^2+f^{-4})/9)$ is not identically zero, in which case we obtain that

$$\frac{f'}{\sqrt{C-(f^2+f^{-4})/9}} = 1.$$

So, by substituting again $\lambda = f^{-3}$ and by integrating we find the desired result. $\square$

Therefore, we have proved the following theorem.

Theorem 3.2. *Let* M *be an* n*-dimensional totally real isotropic submanifold in* $\mathbb{C}P^n(4)$. *If* M *is not totally geodesic at the point* p, *then on a neighbourhood of* p, M *is locally isometric with one of the following manifolds*:

a minimal surface $(n=2)$,

$\frac{\mathrm{SU}(3)}{\mathrm{SO}(3)}$, $\mathrm{SU}(3)$, $\frac{\mathrm{SU}(6)}{\mathrm{Sp}(3)}$, E_6/F_4,

$S^1 \times S^{n-1}(3/2)$, $n > 2$,

$S^1 \times_d S^{n-1}$, *where* d *is defined by* (3.4) *and Lemma* 3.3, $n > 2$.

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Katholieke Universitet Leuven
Department Wiskunde
Celestijnenlaan 200 B
B-3030 Leuven, Belgium

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