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SESQUILINEAR AND QUADRATIC FORMS ON MODULES OVER $*$-ALGEBRAS

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Abstract. We define three new quadratic forms on a module over a $*$-algebra. It is shown that for each quadratic form with a certain property, there exists a sesquilinear form such that both forms are equal to each other. The converse statement is also valid. So far as application is concerned this result enables us to form new characterization formulas for an inner product space if we restrict attention to normed linear spaces.

1. Introduction. In his lecture on Hilbert spaces (in Paris, 1963) I. Halperin raised the following question: Let X be a complex (or real) linear space, if a quadratic functional on X satisfies the parallelogram law and the homogeneity property, does there exist a sesquilinear functional on $X \times X$ which is equal to the given quadratic functional? The positive answer to the question for the complex linear space was given by Kurepa [**6**], and a simplified proof (yet, not short) was done by him [**7**], too. The answer to the same question in the real case is negative provided that the space is one dimensional [**5**]. Vukman [**12**] obtained the same result as Kurepa's, but with different domains and ranges. Indeed, he generalized a functional to a quadratic form on X, a unitary left module over a Banach $*$-algebra with unit, and to a sesquilinear form on $X \times X$. A further generalization of a quadratic form was given in my paper [**8**]. The purpose of this paper is to define three new quadratic forms, and it is shown that each form is equivalent to a sesquilinear form in some sense. As for applications of this result we shall obtain new characterization formulas for an inner product space among normed linear spaces. Some important and well-known characterizations [**1**, **2**, **3**, **4**, **9**, **10**] of this space are special cases of our formulas.

2. The main result. In this paper, except applications in the last section, all algebras and linear spaces will be over the complex number field. **R** will denote the real number field, A a $*$-algebra (not necessarily a Banach $*$-algebra) with unit, and X a linear space which is also a left A-module. Let us call a mapping $B : X \times X \to A$

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an A-sesquilinear form if B is additive in both arguments, $B(ax,y)=aB(x,y)$, and $B(x,ay)=B(x,y)a^*$ for all x and y in X and all a in A [**8**].

Definition. A mapping $Q: X\to A$ is called a generalized A-quadratic form if $Q(ax)=aQ(x)a^*$ for all x in X and all a in A, and any one of the following three identities holds:

$$Q\Big(\sum_{i=1}^{n} a_i x_i\Big)+\sum_{1\le i<j\le n} a_i a_j Q(x_i-x_j)=\Big(\sum_{i=1}^{n} a_i\Big)\Big[\sum_{i=1}^{n} a_i Q(x_i)\Big] \tag{1}$$

for all x_i in X $(i=1,\ldots,n)$, some fixed a_i in $\mathbf{R}$ $(i=1,\ldots,n)$ and at least two of them are nonzero such that $\sum_{i=1}^{n} a_i\neq 0$, and a fixed $n\ge 2$;

$$Q\Big(\sum_{i=1}^{n} a_i x_i\Big)+\sum_{i=1}^{n}\Big[a_i\Big(\Big(\sum_{j=1}^{n} a_j\Big)-2a_i\Big)Q(x_i)\Big]=\sum_{1\le i<j\le n} a_i a_j Q(x_i+x_j) \tag{2}$$

for all x_i in X $(i=1,\ldots,n)$, some fixed a_i in $\mathbf{R}$ $(i=1,\ldots,n)$ and at least three of them are nonzero, and a fixed $n\ge 3$;

$$rQ(sx+ty)+sQ(tx-ry)=(rs+t^2)[sQ(x)+rQ(y)] \tag{3}$$

for all x and y in X, and some fixed nonzero r, s and t in $\mathbf{R}$ with $rs+t^2\neq 0$.

For later use, note that $Q(0)=0$ and $Q(-x)=Q(x)$. Let us call a mapping Q an A-quadratic form if both relations $Q(ax)=aQ(x)a^*$ and $Q(x_1+x_2)+Q(x_1-x_2)=2[Q(x_1)+Q(x_2)]$ are fulfilled. Obviously, it is a special case of (1) when $a_1=a_2=1$ and $a_i=0$ otherwise; of (2) when $x_3=-x_2$, $a_1=a_2=a_3=1$ and $a_i=0$ otherwise; and of (3) when $r=s=t=1$.

Let the mapping $B: X\times X\to A$ be defined in terms of the mapping $Q: X\to A$ as follows:

$$B(x,y)=[Q(x+y)-Q(x-y)+iQ(x+iy)-iQ(x-iy)]/4 \tag{4}$$

for all x and y in X. It was indicated in [**12**, Theorem 7] that if (4) holds and Q is an A-quadratic form, then B is an A-sesquilinear form and $Q(x)=B(x,x)$. One can easily show that the converse also holds. Thus, our main theorem below shows that the notion of A-quadratic form is equivalent to the notion of generalized A-quadratic form if (4) is satisfied.

Theorem 1. *If $B: X\times X\to A$ and $Q: X\to A$ are two mappings, then the following conditions are equivalent*:

(a) *B is an A-sesquilinear form and $Q(x)=B(x,x)$;*

(b) *the identity* (4) *holds and Q is a generalized A-quadratic form* (1);

(c) *the identity* (4) *holds and Q is a generalized A-quadratic form* (2);

(d) *the identity* (4) *holds and Q is a generalized A-quadratic form* (3).

Proof. Firstly, we note that the proof that (a) implies (b), (c) and (d) is a trivial calculation. Secondly, we shall prove the other way round as follows.

(b) $\Rightarrow$ (a): That $B(x,x) = Q(x)$ is clear. In order to prove the relation $B(x+y,z) = B(x,z)+B(y,z)$ for all x, y and z in X, we assume $a_1 \neq 0$ and claim that

$$\begin{aligned} &B\Big(a_1 u + \Big(\sum_{i=2}^n a_i\Big)v, a_1 z\Big) + a_1\Big(\sum_{i=2}^n a_i\Big)B(u-v,z) \\ &\quad = B\Big(\Big(\sum_{i=1}^n a_i\Big)u, a_1 z\Big) = a_1\Big(\sum_{i=2}^n a_i\Big)B\Big(\Big(\sum_{i=1}^n a_i\Big)u/\Big(\sum_{i=2}^n a_i\Big), z\Big) \end{aligned} \tag{5}$$

for any z, u and v in X, and a_i's are as in (1). To prove this, in (1) let $x_1 = u+z$ and $x_i = v$ otherwise; then

$$\begin{aligned} &Q\Big(a_1 u + \Big(\sum_{i=2}^n a_i\Big)v + a_1 z\Big) + a_1\Big(\sum_{i=2}^n a_i\Big)Q(u-v+z) \\ &\quad = \Big(\sum_{i=1}^n a_i\Big)\Big[a_1 Q(u+z) + \Big(\sum_{i=2}^n a_i\Big)Q(v)\Big]. \end{aligned}$$

In the above equation if z is replaced by $-z$, by iz, and by $-iz$, we should get three equations. From all these four equations and with the aid of (4) we should obtain easily an identity expressed in terms of the mapping B, namely

$$B\Big(a_1 u + \Big(\sum_{i=2}^n a_i\Big)v, a_1 z\Big) + a_1\Big(\sum_{i=2}^n a_i\Big)B(u-v,z) = a_1\Big(\sum_{i=1}^n a_i\Big)B(u,z). \tag{6}$$

Note that $B(0,y) = B(x,0) = 0$ for all x and y in X. Let $v = u$ and $v = -a_1 u/\big(\sum_{i=2}^n a_i\big)$ in (6), respectively, then we should get two equations. By substituting these two equations into (6) we have our claim proved.

Now, for any x and y in X let $u = \big(\sum_{i=2}^n a_i\big)y/\big(\sum_{i=1}^n a_i\big)$ in the right hand sides of (5), then

$$B\Big(\Big(\sum_{i=2}^n a_i\Big)y, a_1 z\Big) = a_1\Big(\sum_{i=2}^n a_i\Big)B(y,z). \tag{7}$$

Finally let

$$u = \frac{\big(\sum_{i=2}^n a_i\big)(x+y)}{\sum_{i=1}^n a_i} \qquad \text{and} \qquad v = \frac{\big(\sum_{i=2}^n a_i\big)y - a_1 x}{\sum_{i=1}^n a_i}$$

in (5), and together with (7), we have the additivity in the first argument. Analogously, one can show the additivity in the second argument.

The last step is to verify that $B(ax,y) = aB(x,y)$ and $B(x,ay) = B(x,y)a^*$ for all x and y in X and all a in A. We shall omit the proof of this since it was shown exactly in [**11**]. Incidentally, the proof requires that B be additive in both arguments as we have just proved. This concludes the proof of (b) $\Rightarrow$ (a).

(c) $\Rightarrow$ (a): The proof is similar to that of (b) $\Rightarrow$ (a). We shall prove only that the relation $B(y,z)+B(x,z) = B(x+y,z)$ is valid for all x, y and z in X, and the outlines are as follows. In (2) let $x_1 = u+z$ and $x_i = v$ otherwise, and we may assume, of course, that $a_1 \neq 0 \neq \sum_{i=2}^n a_i$ and $a_1 \neq \sum_{i=2}^n a_i$. It follows that

$$B\Big(a_1 u + \Big(\sum_{i=2}^n a_i\Big)v, a_1 z\Big) + a_1\Big(\Big(\sum_{j=1}^n a_j\Big) - 2a_1\Big)B(u,z) = a_1\Big(\sum_{i=2}^n a_i\Big)B(u+v,z). \tag{8}$$

After two suitable substitutions for v into (8), (8) becomes

$$\begin{aligned} &B\Big(a_1u+\Big(\sum_{i=2}^{n}a_i\Big)v,a_1z\Big)-a_1\Big(\sum_{i=2}^{n}a_i\Big)B(u+v,z)=B\Big(\Big(a_1-\sum_{i=2}^{n}a_i\Big)u,a_1z\Big)\\ &\quad=-a_1\Big(\sum_{i=2}^{n}a_i\Big)B\Big(\Big(-a_1+\sum_{i=2}^{n}a_i\Big)u/\Big(\sum_{i=2}^{n}a_i\Big),z\Big). \end{aligned} \tag{9}$$

Let $u=\big(\sum_{i=2}^{n}a_i\big)y/\big(-a_1+\sum_{i=2}^{n}a_i\big)$ in (9) for any y in X; then

$$B\Big(-\Big(\sum_{i=2}^{n}a_i\Big)y,a_1z\Big)=-a_1\Big(\sum_{i=2}^{n}a_i\Big)B(y,z). \tag{10}$$

Finally, for any x in X let

$$u=\frac{\big(\sum_{i=2}^{n}a_i\big)(x+y)}{-a_1+\sum_{i=2}^{n}a_i} \qquad \text{and} \qquad v=\frac{a_1x+\big(\sum_{i=2}^{n}a_i\big)y}{a_1-\sum_{i=2}^{n}a_i}$$

in (9), and by applying (10) we should obtain the desired equation.

(d) $\Rightarrow$ (a): Here again, we shall outline the proof of the additivity of B in the first argument. Let $x=u+(z/t)$ and $y=v$ in (3); then we should get the relation

$$rB(su+tv,sz/t)+sB(tu-rv,z)=(rs+t^2)sB(u,z/t). \tag{11}$$

After two suitable substitutions for v into (11), (11) becomes

$$\begin{aligned} &rB(su+tv,sz/t)+sB(tu-rv,z)\\ &\quad=rB((rs+t^2)u/r,sz/t)=sB((rs+t^2)u/t,z). \end{aligned} \tag{12}$$

Let $u=ty/(rs+t^2)$ in (12) for any y in X, then

$$rB(ty/r,sz/t)=sB(y,z). \tag{13}$$

Finally, for any x in X let $u=t(x+y)/(rs+t^2)$ and $v=(t^2y-rsx)/r(rs+t^2)$ in (12), and together with (13), we have the desired result and the proof is complete.

It should be noted that the following is a special case of the form (1):

$$Q\Big(\sum_{i=1}^{n}a_ix_i\Big)+\sum_{1\le i<j\le n}a_ia_jQ(x_i-x_j)=0 \tag{1'}$$

for all x_i in X ($i=1,\dots,n$), some fixed a_i in $\mathbf{R}$ ($i=1,\dots,n$) and at least three of them are nonzero such that $\sum_{i=1}^{n}a_i=0$, and a fixed $n\ge 3$. To see this, consider the index $i=1$ to $n+1$ in (1), and let $\sum_{i=1}^{n}a_i=0$, $a_{n+1}=1$ and $x_{n+1}=0$.

The next result is clear.

COROLLARY 1. *Theorem* 1 *still holds if the form* (1) *in the statement* (b) *is replaced by the form* (1′).

Note in particular that in (1′) if $a_1=a_2=1$, $a_3=-2$ and $x_i=0$ ($i=3,\dots,n$), then $Q(x_1+x_2)+Q(x_1-x_2)=2[Q(x_1)+Q(x_2)]$.

3. Applications. The following result is interesting and useful characterization formulas for an inner product space among normed linear spaces.

COROLLARY 2. *Let X be a complex (or real) normed linear space; then the following statements are equivalent:*

(a) *X is an inner product space;*

(b) *The norm in X satisfies the condition:*

$$\Big\| \sum_{i=1}^{n} a_i x_i \Big\|^2 + \sum_{1\le i<j\le n} a_i a_j \|x_i - x_j\|^2 = \Big(\sum_{i=1}^{n} a_i\Big)\Big(\sum_{i=1}^{n} a_i \|x_i\|^2\Big)$$

for all x_i in X $(i = 1, \dots, n)$, some fixed a_i in $\mathbf{R}$ $(i = 1, \dots, n)$ and at least two of them are nonzero such that $\sum_{i=1}^{n} a_i \ne 0$, and a fixed $n \ge 2$;

(c) *The norm in X satisfies the condition:*

$$\Big\| \sum_{i=1}^{n} a_i x_i \Big\|^2 + \sum_{1\le i<j\le n} a_i a_j \|x_i - x_j\|^2 = 0$$

for all x_i in X $(i = 1, \dots, n)$, some fixed a_i in $\mathbf{R}$ $(i = 1, \dots, n)$ and at least three of them are nonzero such that $\sum_{i=1}^{n} a_i \ne 0$, and a fixed $n \ge 3$;

(d) *The norm in X satisfies the condition:*

$$\Big\| \sum_{i=1}^{n} a_i x_i \Big\|^2 + \sum_{i=1}^{n} \Big[a_i \Big(\Big(\sum_{j=1}^{n} a_j \Big) - 2a_i \Big) \|x_i\|^2 \Big] = \sum_{1\le i<j\le n} a_i a_j \|x_i + x_j\|^2$$

for all x_i in X $(i = 1, \dots, n)$, some fixed a_i in $\mathbf{R}$ $(i = 1, \dots, n)$ and at least three of them are nonzero, and a fixed $n \ge 3$;

(e) *The norm in X satisfies the condition:*

$$r\|sx + ty\|^2 + s\|tx - ry\|^2 = (rs + t^2)(s\|x\|^2 + r\|y\|^2)$$

for all x and y in X, some fixed nonzero r, s and t in $\mathbf{R}$ such that $rs + t^2 \ne 0$.

In the statement (b), (c), (d) and (e) the inner product is defined as usual by

$$(x|y) = \tfrac{1}{4}[\|x+y\|^2 - \|x-y\|^2 + i\|x+iy\|^2 - i\|x-iy\|^2] \quad \text{and}$$
$$(x,y) = \tfrac{1}{4}[\|x+y\|^2 - \|x-y\|^2]$$

for the complex and real spaces, respectively.

Proof. Obviously, this is a special case of Theorem 1 when X is a complex (or real) normed linear space and A is the field of complex (or real) numbers. In fact, the mapping Q is a square norm and the mapping B is an inner product, i.e., $Q(x) = \|x\|^2$ and $B(x,y) = (x|y)$ (or $= (x,y)$) the usual inner product of x and y.

In closing, it might be worth while to remark that many important and well-known characterizations of an inner product space are just special cases of Corollary 2. The following are some examples:

(i) The condition that $\|x+y\|^2 + \|x-y\|^2 = 2(\|x\|^2 + \|y\|^2)$ for all x and y in X [**4**] may be obtained from (b), (c), (d) or (e) in Corollary 2.

(ii) Let $\|x\| = \|y\|$, $r = -s = -a$ and $t = b$ in (e); then $\|ax + by\| = \|bx + ay\|$, the main result in [**2**]. Our proof is simpler and different from Ficken's.

(iii) Let $\|x\| = \|y\|$, $s = -r = a$ and $t = 1$ in (e); then $\|ax + y\| = \|x + ay\|$ [**9**].

(iv) Let $\sum_{i=1}^n x_i = 0$ for $n \geq 3$ and $a_i = 1$ $(i = 1, \dots, n)$ in (b); then $\sum_{i \neq j} \|x_i - x_j\|^2 = 2n \sum_{i=1}^n \|x_i\|^2$ [**9**].

(v) Let $r = (1-a)/a$, $s = b/(1-b)$ for $0 < a, b < 1$ and $t = 1$ in (e); then

$$\begin{aligned} a(1-a)\|bx + (1-b)y\|^2 + b(1-b)\|ax + (1-a)y\|^2 \\ = (a + b - 2ab)[ab\|x\|^2 + (1-a)(1-b)\|y\|^2] \end{aligned}$$

for all x and y in X [**1**].

(vi) The statement (c) in Corollary 2 is precisely the main result in [**10**]. Our proof is direct, and does not depend on the Jordan-Neumann condition.

(vii) In (d) let $a_1 = a_2 = a_3 = 1$, and $a_i = 0$ otherwise; then

$$\|x_1 + x_2 + x_3\|^2 + \|x_1\|^2 + \|x_2\|^2 + \|x_3\|^2 = \|x_1 + x_2\|^2 + \|x_2 + x_3\|^2 + \|x_3 + x_1\|^2$$

for all x_1, x_2 and x_3 in X [**3**].

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