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ON THE URYSOHN INTEGRAL EQUATION IN LOCALLY CONVEX SPACES

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Abstract. This paper contains an existence theorem for the Urysohn integral equation in locally convex spaces. In the proof of this theorem we employ a modified version of Mönch's fixed point theorem and measures of noncompactness.

1. Introduction. By repeating Mönch's argument from the proof of Theorem 2.2 of [**4**] and by using the Schauder-Tychonoff theorem instead of the Schauder theorem, we can prove the following fixed point theorem.

THEOREM 1. *Let D be an open subset of a quasicomplete locally convex space X, $0 \in D$, and let G be a continuous mapping of $\overline{D}$ into X. If the implication*

$$V \subset \overline{\operatorname{conv}}(\{0\} \cup G(V)) \quad \Longrightarrow \quad V \text{ is relatively compact}$$

holds for every countable subset V of $\overline{D}$, and G satisfies the boundary condition

$$x \in \overline{D},\ 0 < \alpha < 1,\ x = \alpha G(x) \quad \Longrightarrow \quad x \notin \partial D\,,$$

then G has a fixed point in $\overline{D}$.

Let $T = [0, a]$ and let W be an open subset of a quasicomplete locally convex space E. In Section 2 we shall apply Theorem 1 to obtain an existence theorem for continuous solutions of the Urysohn integral equation

$$x(t) = g(t) + \lambda \int_T f(t, s, x(s))\, ds\,, \tag{1}$$

where f is a bounded continuous function from $T \times T \times W$ into E and g is a continuous function from T into W. Next, in Section 3, by applying Lemma of [**6**] we shall show that the set of all continuous solutions of Volterra integral equation

$$x(t) = g(t) + \int_0^t f(t, s, x(s))\, ds \tag{2}$$

is a continuum in the corresponding function space.

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2. An existence theorem. Let P be a family of continuous seminorms generating the topology of E. For any $p \in P$ and for any bounded subset A of E denote by $\beta_p(A)$ the infimum of all $\varepsilon > 0$ for which there exists a finite subset Q of E such that $A \subset Q + B_p(\varepsilon)$, where $B_p(\varepsilon) = \{x \in E : p(x) \le \varepsilon\}$. The family $(\beta_p(A))_{p\in P}$ is called the Hausdorff measure of noncompactness of A (for properties see [**5**]). Denote by $C(T,E)$ the space of continuous functions $T \to E$ with the topology of uniform convergence. For any subset H of $C(T,E)$ put $H(t) = \{u(t) : u \in H\}$. The following has been proved in [**8**]:

LEMMA. *If the space E is separable, then for any bounded countable subset H of $C(T,E)$ the function $t \to \beta_p(H(t))$ is measurable on T and*

$$\beta_p\left(\left\{\int_T x(t)\,dt : x \in H\right\}\right) \le \int_T \beta_p(H(t))\,dt\ .$$

Let Ω denote the family of all open, balanced and convex neighbourhoods of 0 in E. We assume that

(3) for each $U \in \Omega$ there exists an $\varepsilon > 0$ such that $f(t,s,x) - f(r,s,x) \in U$ for $x \in W$ and $s,t,r \in T$ such that $|t-r| < \varepsilon$.

THEOREM 2. *Assume that for each $p \in P$ there exists a continuous function $K_p : T \times T \to \mathbf{R}_+$ such that*

$$\beta_p(f(t,s,Y)) \le K_p(t,s)\beta_p(Y) \tag{4}$$

for $t,s \in T$ and for each bounded subset Y of W. Moreover, assume that there is an $r_0 > 0$ such that for each $p \in P$ the spectral radius $r(\widetilde{K}_p)$ of the integral operator $\widetilde{K}_p$, defined by

$$\widetilde{K}_p u(t) = \int_T K_p(t,s)u(s)\,ds \qquad (u \in C(T,\mathbf{R}),\ t \in T)$$

is less than r_0. Then there exists a positive number η such that for each $\lambda \in \mathbf{R}$ with $|\lambda| < \eta$ the equation (1) has at least one continuous solution.

Proof. As W is open and g is continuous, we can choose a set B of the form $B = \{x \in E : p_i(x) \le b, i = 1,\ldots,m\}$, where, $p_1,\ldots,p_m \in P$, such that $g(t) + B \subset W$ for $t \in T$. From the boundedness of f it follows that there exists a $\rho > 0$ such that $[-\rho,\rho]\overline{\mathrm{conv}}\, f(T \times T \times W) \subset B$. Let $\eta = \min(\rho/\operatorname{mes} T, 1/2r_0)$. Fix $\lambda \in \mathbf{R}$ with $|\lambda| < \eta$. Put

$$H = \{u \in C(T,E) : u(t) - g(t) \in B \ \text{ for } t \in T\}$$

and

$$F(x)(t) = g(t) + \lambda \int_T f(t,s,x(s))\,ds \qquad (x \in H,\ t \in T).$$

As

$$F(x)(t) - g(t) \in [-|\lambda|, |\lambda|] \operatorname{mes} T \, \overline{\operatorname{conv}} \, f(T \times T \times W)$$
$$\subset [-\rho, \rho] \, \overline{\operatorname{conv}} \, f(T \times T \times W) \subset B \qquad \text{for } x \in H,$$

we see that F maps H into H. Moreover, from (3) it is clear that the set $F(H)$ is equiuniformly continuous. By Lemma 2 of [**7**] for any $u \in H$ and $U_1 \in \Omega$ there exists a $U_2 \in \Omega$ such that

$$f(t,s,x(s)) - f(t,s,u(s)) \in U_1 \qquad \text{for } \ t, s, \in T,$$

whenever $x \in H$ and $x(t) - u(t) \in U_2$ for all $t \in T$. From this we deduce that F is continuous.

Put $G(x) = F(x+g) - g$ for $x \in D = \{u \in C(T,E) : u(t) \in B \ \text{ for } t \in T\}$. Then G is a continuous mapping $D \to D$. Now we shall show that G satisfies the assumptions of Theorem 1.

Assume that $x \in D$, $x = \alpha G(x)$ and $0 < \alpha < 1$, and suppose that $x \in \partial D$. Then there exist a $t \in T$ and an i, $1 \le i \le m$, such that $p_i(x(t)) = b$. As $G(x) \in D$, we have $b = p_i(x(t)) = \alpha p_i(G(x)(t)) \le \alpha b < b$, which is impossible.

Assume now that $V = \{u_n : n \in \mathbf{N}\}$ is a countable subset of D such that

$$V \subset \overline{\operatorname{conv}}(G(V) \cup \{0\}). \tag{5}$$

Then

$$V(t) \subset \overline{\operatorname{conv}}(G(V)(t) \cup \{0\}) \qquad \text{for } \ t \in T. \tag{6}$$

Let (t_n) be a dense sequence in T, and let Z be the closed linear hull of the set

$$\left\{ g(t_i), u_n(t_i), f(t_i, t_j, u_n(t_k) + g(t_k)) : i,j,k,n \in \mathbf{N} \right\}.$$

Then Z is a separable quasicomplete locally convex subspace of E, and $g(t) \in Z$, $f(t,s,u_n(s)+g(s)) \in Z$, $u_n(t) \in Z$, $G(u_n)(t) \in Z$ for $t, s \in T$ and $n \in \mathbf{N}$.

For any bounded subset A of Z and $p \in P$, denote by $\beta_p^z(A)$ the infimum of all $\varepsilon > 0$ for which there exists a finite subset Q of Z such that $A \subset Q + B_p(\varepsilon)$. Since the set $G(V)$ is equiuniformaly continuous, from (5) it follows that the function $t \to \beta_p(V(t))$ is continuous. It is clear from (4) that

$$\beta_p^z(\{f(t,s,u_n(s)+g(s)) : n \in \mathbf{N}\}) \le 2\beta_p(\{f(t,s,u_n(s)+g(s)) : n \in \mathbf{N}\})$$
$$\le 2K_p(t,s)\beta_p(\{u_n(s)+g(s) : n \in \mathbf{N}\}) = 2K_p(t,s)\beta_p(V(s)).$$

Hence, by (6) and Lemma, we get

$$\begin{aligned}
\beta_p(V(t)) &\le \beta_p(G(V)(t)) \le \beta_p^z(G(V)(t)) \\
&= \beta_p^z\left(\left\{ \lambda \int_T f(t,s,u_n(s)+g(s))\, ds : n \in \mathbf{N} \right\}\right) \\
&\le |\lambda| \int_T \beta_p^z(\{f(t,s,u_n(s)+g(s)) : n \in \mathbf{N}\})\, ds \\
&\le 2|\lambda| \int_T K_p(t,s)\beta_p(V(s))\, ds \qquad \text{for } t \in T.
\end{aligned}$$

As $2|\lambda|r(\widetilde{K}_p) \leq 2|\lambda|r_0 < 1$, this implies that $\beta_p(V(t)) = 0$ for $t \in T$ and $p \in P$. hence for any $t \in T$ the set $V(t)$ is relatively compact in E. By Ascoli's theorem [**3**, Th. 7.17] we deduce from this that V is relatively compact in $C(T, E)$. Now we can apply Theorem 1 which yields the existence of $u \in D$ such that $u = G(u)$. Obviously $x = u - g \in H$ and $x = F(x)$, so that x is a continuous solution of (1).

3. A Kneser-Hukuhara theorem. Consider now the equation (2). Let us recall that a function $h : T \times T \times \mathbf{R}_+ \to \mathbf{R}_+$ is called a Kamke function if h satisfies the Caratheodory conditions and, for each $0 < d \leq a$, the function $u = 0$ is the unique nonnegative continuous solution of the inequality

$$u(t) \leq \int_0^t h(t, s, u(s))\, ds \qquad \text{on} \quad [0, d].$$

By arguing similarly as in the proof of Theorem 2 and by applying Lemma from [**6**], we can prove the following

THEOREM 3. *Assume that for any $p \in P$ there exists a function $(t, s, u) \to h_p(t, s, u)$ such that $2h_p$ is a Kamke function, h_p is nondecreasing in u and*

$$\beta_p(f(t, s, X)) \leq h_p(t, s, \beta_p(X))$$

for $t, s \in T$ and for each bounded subset X of E. Then there exists an inerval $J = [0, d]$ such that the set of all continuous solutions $x : J \to E$ of (2), considered as a subset of $C(J, E)$, is nonempty, compact and connected.

Let us remark that the result above generalizes Theorem 1 of [**6**].

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