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A NEW PROOF OF PREŠIĆ'S THEOREM ON FINITE EQUATIONS

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Abstract. Applying some kind of algebraic structure Prešić determined in [**2**] the formulas of all general reproductive solutions of finite equation. We give a shorter proof of Prešić's theorem.

Definition 1. Let E be a given non-empty set and eq be a given unary relation of E. A formula $x = \varphi(t)$, where $\varphi : E \to E$ is a given function, represents a general reproductive solution of the x-equation $\mathrm{eq}(x)$ if and only if

$$(\forall t)\, \mathrm{eq}(\varphi(t)) \wedge (\forall t)(\mathrm{eq}(t) \implies t = \varphi(t)).$$

Let the set R be defined as follows $t \in R \iff \mathrm{eq}(t) \qquad (t \in E)$.

LEMMA [**1**]. *The formula $x = \varphi(t)$ represents a general reproductive solution of* $\mathrm{eq}(x)$ *if and only if* $(\forall t)(t \in R \implies t = \varphi(t)) \wedge (\forall t)(t \notin R \implies \varphi(t) \in R)$.

Let $Q = \{q_0, q_1, \ldots, q_m\}$ be a given set of $m+1$ elements and $S = \{0, 1\}$. Define an operation x^y by

$$x^y = \begin{cases} 1, & \text{if } x = y \\ 0, & \text{otherwise} \end{cases} \qquad (x, y \in Q \cup S)$$

The operations $+$ and $\circ$ are described by the following tables

$$\begin{array}{c|cc} + & 0 & 1 \\ \hline 0 & 0 & 1 \\ 1 & 1 & 1 \end{array} \qquad\qquad \begin{array}{c|cc} \circ & 0 & 1 \\ \hline 0 & 0 & 0 \\ 1 & 0 & 1 \end{array}$$

Extending these operations to the partial operations of $Q \cup S$ by

$$x + 0 = x, \qquad 0 + x = x, \qquad x \circ 0 = 0, \qquad 0 \circ x = 0, \qquad x \circ 1 = x, \qquad 1 \circ x = x,$$

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Prešić considered the following x-equation

$$s_0 \circ x^{q_0} + s_1 \circ x^{q_1} + \cdots + s_m \circ x^{q_m} = 0, \tag{1}$$

where $s_i \in \{0,1\}$ are given elements and $x \in Q$ is unknown. It is obvious that the equation (1) is possible if and only if $s_0 \circ s_1 \circ \ldots \circ s_m = 0$.

In the sequal $\circ$ will be omitted.

Let $(r_0, \ldots, r_m) \in \{0,1\}^{m+1}$. The set $Z(r_0, \ldots, r_m)$, zero-set of $(r_0, \ldots, r_m)$, is defined as follows:

$$q_i \in Z(r_0, \ldots, r_m) \iff r_i = 0 \qquad (i = 0, 1, \ldots, m), \tag{2}$$

i.e. $Z(r_0, \ldots, r_m)$ is the solution set of the equation $r_0 x^{q_0} + r_1 x^{q_1} + \cdots + r_m x^{q_m} = 0$.

Theorem 1 (Prešić [**2**]). *If the equation*

$$s_0 x^{q_0} + s_1 x^{q_1} + \cdots + s_m x^{q_m} = 0 \tag{3}$$

is possible, then the formula $x = A(t)$ (t is any element of Q) represents a general reproductive solution of the equation (3) *if and only if*

$$A(t) = \sum_{k=0}^{n} \Bigg(q_k s_k^0 + \sum_{a_k=1,\, a_0 \ldots a_m = 0} F_k(a_0, \ldots, a_m) s_0^{a_0} \ldots s_m^{a_m} \Bigg) t^{q_k} \tag{4}$$

assuming that the coefficients $F_k(a_0, \ldots, a_m) \in Q$ satisfy the condition $F_k(a_0, \ldots$ $\ldots, a_m) \in Z(a_0, \ldots, a_m)$.

Proof. Let $A(t)$ be of the form (4). For arbitrary $t \in Q$ there is a $k \in \{0, 1, \ldots, m\}$ such that $t = q_k$. If $q_k \in Z(s_0, s_1, \ldots, s_m)$, then the formula $x = A(t)$ gives $x = q_k$, because of (2). If $q_k \notin Z(s_0, s_1, \ldots, s_m)$, then the formula $x = A(t)$ becomes $x = F_k(s_0, s_1, \ldots, s_m)$. We also have $F_k(s_0, s_1, \ldots, s_m) \in Z(s_0, s_1, \ldots, s_m)$. Therefore the formula $x = A(t)$ represents a general reproductive solution of (3), by the lemma.

Let the formula $x = A(t)$ represent a general reproductive solution of (3). If we write the function A as

$$A = \begin{pmatrix} q_0 & q_1 & \cdots & q_m \\ q_{i_0} & q_{i_1} & \cdots & q_{i_m} \end{pmatrix},$$

then $\{q_{i_0}, q_{i_1}, \ldots, q_{i_m}\} = Z(s_0, s_1, \ldots, s_m)$. If $q_k \notin Z(s_0, s_1, \ldots, s_m)$, then we determine $F_k(s_0, s_1, \ldots, s_m)$ in the following way:

$$F_k(s_0, s_1, \ldots, s_m) = q_{i_k} \qquad (k = 0, 1, \ldots, m). \tag{5}$$

Since A represents a general reproductive solution of (3) we have

$$q_k \in Z(s_0, s_1, \ldots, s_m) \Rightarrow A(q_k) = q_k. \tag{6}$$

Since $q_k \in Z(s_0, s_1, \dots, s_m) \Rightarrow s_k = 0 \Rightarrow q_k s_k^0 = q_k$ and since the sum $\sum_{a_k=1,\, a_0 \dots a_m=0} F_k(a_0, \dots, a_m) s_0^{a_0} \dots s_m^{a_m}$ reduces to $F_k(s_0, s_1, \dots, s_m)$, A can be written as

$$A(t) = \sum_{k=0}^{n} \left(q_k s_k^0 + \sum_{a_k=1,\, a_0 \dots a_m=0} F_k(a_0, , \dots, a_m) s_0^{a_0} \dots s_m^{a_m} \right) t^{q_k}$$

because of (5), (6) and the lemma.

REFERENCES

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