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CONTROL OF A GAUSSIAN PROCESS BY RAREFYING ITS INNOVATION PROCESS

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Abstract. We generalized the model of the rarefaction of a continuous Gaussian process, introduced in [**1**]. It is more suitable to consider this proposed rarefaction as a control of the given process.

For the sake of simplicity we consider the mean-square continuous Gaussian process $\{\xi(t), t \geq 0\}$, $E\xi(t) = 0$, with the multiplicity $N = 1$ and of the spectral type equivalent to the spectral type of a Wiener process $\{W(t), t \geq 0\}$. The proper canonical (or Hida-Cramér) representation of $\{\xi(t)\}$ is

$$\xi(t) = \int_0^t g(t,u)W(du), \qquad g(t,\,\cdot\,) \in \mathcal{L}_2(du). \tag{1}$$

Let $\mathcal{F}_t(\eta)$ be the σ-field generated by $\{\eta(t), u \leq t\}$. In the representation (1) one has $\mathcal{F}_t(\xi) = \mathcal{F}_t(W)$, $t > 0$. Also, as $\{\xi(t)\}$ is Gaussian, for $s < t$ we have

$$E\big(\xi(t) \mid \mathcal{F}_s(\xi)\big) = \int_0^s g(t,u)W(du).$$

In [**1**] we defined the rarefied process $\{W^*(t), t \geq 0\}$ of $\{W(t), t \geq 0\}$ by

$$W^*(t) = \int_0^t I\big(u, W(u), (du)^{-1/2}W(du)\big)\, W(du), \tag{2}$$

where I is a measurable indicator function. In this note we generalize (2) in the following way.

Let $I(u)$ be some Riemann integrable indicator function, measurable with respect to $\mathcal{F}_u(\xi)$ for $u \geq 0$. We define the rarefied process $\{W^*(t), t \geq 0\}$ of $\{W(t), t \geq 0\}$ by

$$W^*(t) = \int_0^t I(u)W(du). \tag{3}$$

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The integral (3) is the mean-square limit of the integral sum

$$\sum_0^t I(u_k)W(\Delta_k), \qquad W(\Delta_k) = W(u_{k+1}) - W(u_k)$$

over all finite partitions $\{\Delta_k\}$ of $(0,t]$.

PROPOSITION 1. *The process* $\{W^*(t),\, t \geq 0\}$ *is a martingale.*

Proof. By the smoothing property of a conditional expectation, for $s < t$ we have

$$\begin{aligned} E\big(W^*(t) \mid \mathcal{F}_s(W^*)\big) &= E\big(E\big(W^*(t) \mid \mathcal{F}_s(\xi)\big) \mid \mathcal{F}_s(W^*)\big) \\ &= E\big(E\big(W^*(t) - W^*(s) + W^*(s) \mid \mathcal{F}_s(\xi)\big) \mid \mathcal{F}_s(W^*)\big) \\ &= E\big(E\big(W^*(t) - W^*(s) \mid \mathcal{F}_s(\xi)\big) \mid \mathcal{F}_s(\xi)\big) + W^*(s). \end{aligned}$$

By the continuity property of a conditional expectation,

$$\begin{aligned} E\big(W^*(t) - W^*(s) \mid \mathcal{F}_s(\xi)\big) &= E\bigg(\lim_{\Delta\to 0}\sum_s^t I(u_k)W(\Delta_k) \mid \mathcal{F}_s(\xi)\bigg) \\ &= \lim_{\Delta\to 0}\sum_s^t E\big(I(u_k)W(\Delta_k) \mid \mathcal{F}_s(\xi)\big), \end{aligned}$$

where $\Delta = \max\{\Delta_k\}$. As $I(u_k)$ is measurable with respect to $\mathcal{F}_{u_k}(\xi)$ and $W(\Delta_k)$ is independent of $\mathcal{F}_{u_k}(\xi)$, it follows that

$$\begin{aligned} E\big(I(u_k)W(\Delta_k) \mid \mathcal{F}_s(\xi)\big) &= E\big(E\big(I(u_k)W(\Delta_k) \mid \mathcal{F}_{u_k}(\xi)\big) \mid \mathcal{F}_s(\xi)\big) \\ &= E\big(I(u_k)E\big(W(\Delta_k) \mid \mathcal{F}_{u_k}(\xi)\big) \mid \mathcal{F}_s(\xi)\big) = E\big(I(u_k)\cdot 0 \mid \mathcal{F}_s(\xi)\big) = 0. \end{aligned}$$

Finally,

$$E\big(W^*(t) \mid \mathcal{F}_s(W^*)\big) = W^*(s). \ \square$$

The rarefied process $\{\xi^*(t), t \geq 0\}$ of $\{\xi(t), t \geq 0\}$ is defined by

$$\xi^*(t) = \int_0^t g(t,u)W^*(du). \tag{4}$$

Let $\mathcal{H}_t(\eta)$ be the mean-square linear closure of $\{\eta(u), u \leq t\}$. Repeating the arguments in [**1**] we conclude that $\mathcal{H}_t(W^*) = \mathcal{H}_t(\xi^*)$, so it follows that $\mathcal{F}_t(W^*) = \mathcal{F}_t(\xi^*)$ for $t > 0$.

We remark that $\{W^*(t)\}$ and $\{\xi^*(t)\}$ are not necessarily Gaussian processes. Nevertheless, one has the following

PROPOSITION 2. *For* $s < t$:

$$E\big(\xi^*(t) \mid \mathcal{F}_s(\xi^*)\big) = \int_0^s g(t,u)W^*(du).$$

Proof. This relation follows from the fact that $\{W^*(t)\}$ is a martingale:

$$\begin{aligned}
E\big(\xi^*(t) \mid \mathcal{F}_s(\xi^*)\big) &= E\bigg(\int_0^t g(t,u)W^*(du) \,\bigg|\, \mathcal{F}_s(\xi^*)\bigg) \\
&= E\bigg(\int_0^t g(t,u)W^*(du) \,\bigg|\, \mathcal{F}_s(W^*)\bigg) \\
&= \int_0^s g(t,u)W^*(du) + E\bigg(\int_s^t g(t,u)W^*(du) \,\bigg|\, \mathcal{F}_s(W^*)\bigg),
\end{aligned}$$

so the linear and non-linear predictions coincide. $\square$

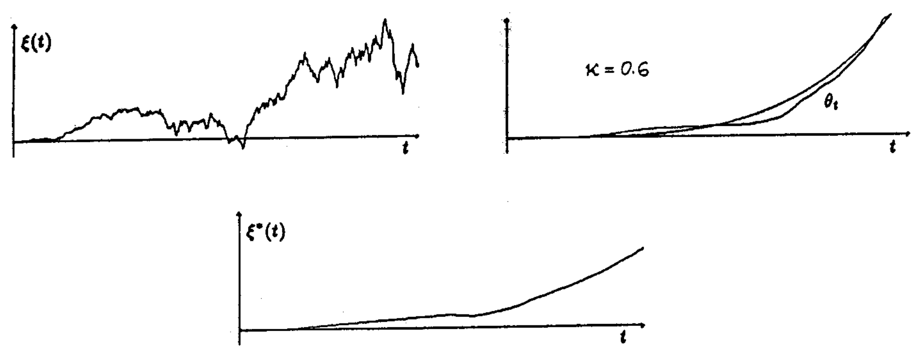
The control of $\{\xi(t)\}$ is proceeded by some functional θ_t measurable with respect to $\mathcal{F}_t(\xi)$. The rule is the following: if θ_u is outside of some "admissible" measurable set S_u at the moment u, then the "infinitesimal innovation" $W(du)$ of $\{\xi(t)\}$ is erased. In other words, $I(u)$ is the indicator of the event $\{\theta_u \in S_u\}$.

Example. Let $\theta_s = \displaystyle\int_0^s \xi^2(u)\,du$ be the average energy on the interval $(0,s]$ and $S_u = (0, kE(\theta_u))$ for some fixed $k > 0$. Let $\{\xi(t), t \geq 0\}$ be defined by

$$\xi(t) = \int_0^t (t+u)W(du). \tag{5}$$

It is easy to prove that (5) is the proper canonical representation. Also we have that $E(\theta_s) = (7/24)s^4$.

In the figures below we present a discretization of one sample path of $\{\xi(t)\}$, θ_t and $\{\xi^*(t)\}$.



REFERENCE

[1] Z. Ivković, P. Peruničić, *Transformation of a continuous Gaussian process by rarefying its innovation process*, Теориј веројтностей и ее применениј **30** (1988), 794–800.

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