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RANDOM SUM LIMIT THEOREMS FOR NONIDENTICALLY DISTRIBUTED RANDOM VARIABLES

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Abstract. Two limit theorems for sums with random indices are proved, one related to triangular arrays, the other to normed sums. No conditions concerning moments are imposed.

Starting from the classical work of Robbins [**1**] and Anscombe [**2**], many versions of the random index limit theorem for sums of random variables were proved. However, not many of them treated the case when random variables are not necessarily identically distributed. Among those that did, are theorems from [**3**], [**4**], [**5**], [**6**], [**7**]. Here we shall be specially concerned with the results from [**3**] and [**4**]. We shall try to weaken conditions imposed in the limit theorems from [**3**] and [**4**], in particular — conditions concerning moments of random variables.

Suppose that

$$\begin{array}{l} X_{11},\dots,X_{1k},\dots \\ \dots\dots\dots\dots\dots \\ X_{n1},\dots,X_{nk},\dots \\ \dots\dots\dots\dots\dots \end{array} \tag{1}$$

are infinite sequences or row-wise independent and not necessarily identically distributed random variables, $P\{X_{nk} \leq x\} = F_{nk}(x)$, $n \geq 1$, $k \geq 1$, k_n is a positive integer-valued sequence such that $k_n \to \infty$ as $n \to \infty$; ν_n is a sequence of positive integer-valued random variables, independent of X_{nk}, $k \geq 1$.

Kruglov [**3**] proved the following theorem:

Theorem. *Suppose that the following conditions hold*:

(A) $EX_{nk} = 0$, $DX_{nk} = \sigma_n^2 < +\infty$ *for* $n \geq 1$, $k \geq 1$; $\lim_{n\to\infty} \sigma_n^2 = 0$;

(B) $\lim_{n\to\infty} P\{\nu_n/k_n \leq x\} = A(x)$, $k_n = [\sigma_n^{-2}]$

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where A is a proper probability distribution and (here and in the sequel) $[x]$ denotes the integer part of x;

(C) $\lim_{n\to\infty} P\left\{ \sum_{j=lk_n+1}^{(l+1)k_n} X_{nj} \le x \right\} = \Phi(x).$

for every $l = 0, 1, 2, \ldots,$ where Φ is $\mathcal{N}(0,1)$ probability distribution.

Then we have

$$\lim_{n\to\infty} P\left\{ \sum_{k=1}^{\nu_n} X_{nk} \le x \right\} = G(x),$$

and the characteristic function ψ of G satisfies:

$$\psi(t) = \int_0^\infty \exp(-yt^2/2)\, dA(y).$$

Here we want to prove a limit theorem for sums of infinitely increasing number of random variables (1), but without the assumptions about the convergence to Normal distribution (conditions A and C from Kruglov's theorem). We remain in the setting of the classical summation theory because we assume that random variables (1) satisfy the following u.a.n. (uniformly asymptotically negligible) condition: for $l = 0, 1, 2, \ldots$

$$\lim_{n\to\infty} \max_{lk_n+1\le k\le (l+1)k_n} P\{|X_{nk}| \ge \varepsilon\} = 0. \tag{2}$$

THEOREM 1. *Suppose the following conditions are fulfilled:*

(A$'$) *For each $l = 0, 1, 2, \ldots$ and every sequence of sets $J_n \subset \{lk_n+1, \ldots, (l+2)k_n\}$,* card $J_n = k_n$,

$$\lim_{n\to\infty} P\left\{ \sum_{j\in J_n} X_{nj} \le x \right\} = F(x)$$

holds at every continuity point of a proper probability distribution F;

(B) $\lim_{n\to\infty} P\{\nu_n/k_n \le x\} = A(x),\ A(0) = 0,$

where A is a proper probability distribution.

Then we have

$$\lim_{n\to\infty} P\left\{ \sum_{k=1}^{\nu_n} X_{nk} \le x \right\} = G(x),$$

where

$$\psi(t) = \int_0^\infty (\varphi(t))^y\, dA(y)$$

and ψ and φ are characteristic functions of G and F, respectively.

Obviously, the probability distribution F from the condition A$'$ could be any infinitely divisible probability distribution.

We shall show that Kruglov's theorem is a special case of the Theorem 1. In order to prove this, it is sufficient to prove that Kruglov's conditions A and C imply the condition A′ of the Theorem 1. So, let us suppose that the conditions A and C are fulfilled and let us recall the following theorem [**8**, p. 314], giving necessary and sufficient conditions for the convergence of sums of infinitely increasing number of independent random variables to the Normal law.

THEOREM. *Let X_{nk} be a sequence of row-wise independent random variables such that the sequence of sums*

$$\sum_{k=1}^{k_n} X_{nk}$$

converges weakly to some nondegenerate random variable. Then the limit law is Normal, and the u.a.n. condition is satisfied if and only if

$$\lim_{n\to\infty} \sum_{k=1}^{k_n} P\{|X_{nk}| \geq \varepsilon\} = 0$$

for every fixed $\varepsilon > 0$.

The uniform asymptotic negligibility of the random variables satisfying condition A could be easily obtained using Chebyshev's inequality. Together with the condition C, this implies the validity of conditions of the above theorem and therefore, for every fixed l, $l = 0, 1, 2, \dots$ and $J_n \subset \{lk_n+1, \dots, (l+2)k_n\}$, $\operatorname{card} J_n = k_n$, we have

$$\sum_{j\in J_n} P\{|X_{nj}| \geq \varepsilon\} \leq \sum_{k=lk_n+1}^{(l+2)k_n} P\{|X_{nk}| \geq \varepsilon\} \to 0$$

so that $\sum_{j\in J_n} X_{nj}$ is the sum tending to $\mathcal{N}(0,1)$ law as $n \to \infty$, which means that the condition A′ from the Theorem 1 is fulfilled too, with the distribution F being Normal.

The second transfer theorem that we shall prove is related to the class L of distributions. Namelly let X_n, $n = 1, 2, \dots$ be a sequence of independent random variables for which the following u.a.n. condition holds:

$$\lim_{n\to\infty} \max_{1\leq k\leq n} P\{|X_k| \geq b_n\varepsilon\} = 0,$$

where b_n is a sequence of positive real numbers. Set $S_n = \sum_{k=1}^{n} X_k$ for $n \geq 1$. L is the class of distributions which are the weak limits of distributions of the sums

$$b_n^{-1}(S_n - a_n), \qquad n \geq 1, \tag{3}$$

where $b_n \geq 0$ and a_n are suitably chosen constants (in terms of characteristic functions (3) becomes

$$\exp(-isa_nb_n^{-1}) \prod_{j=1}^{n} f_j(sb_n^{-1}), \qquad n \geq 1,$$

where f_j is the characteristic function of X_j).

THEOREM 2. *Suppose the following conditions hold*:

A) *There exists a proper probability distribution* F *such that*

$$\lim_{n\to\infty} P\{b_n^{-1}(S_n - a_n) \le x\} = F(x)$$

at all continuity points of F;

B) *For each* $0 < t \le 1$, *the following limit exists*:

$$\lim_{n\to\infty} P\{b_n^{-1}(S_{[nt]} - a_n) \le x\} = G_t(x);$$

C) ν_n, $n = 1, 2, \dots$, *is a sequence of positive, integer-valued random variables, independent of* X_k, $k = 1, 2, \dots$, *such that*

$$\lim_{n\to\infty} P\{\nu_n/n \le x\} = Q(x)$$

weakly to a proper probability distribution Q.

Then we have

$$\lim_{n\to\infty} P\{b_n^{-1}(S_{\nu_n} - a_n) \le x\} = H(x),$$

where the characteristic function h *of the probability distribution* H *safisfies*

$$h(s) = \int_0^\infty \exp\bigl(isA(y)\bigr) f\bigl(B(y)s\bigr)\, dQ(y). \tag{4}$$

Here f *is the characteristic function of* F *and functions* A *and* B *are given by*

$$A(y) = C(1 - y^\alpha), \quad B(y) = y^\alpha, \qquad y > 0,\ \alpha > 0,\ C = \mathit{const.}$$

In the article [**4**] Mogyoródi proved a transfer theorem for the normed sums but, instead of the condition B, he supposed that $EX_n = 0$, D^2X_n exists for $n = 1, 2, \dots$, and that the sequence b_n from (3) is regularly varying, i.e. that for some positive α, $b_n = n^\alpha L(n)$, where $L(n)$ is a slowly varying sequence (namely the sequence satisfying $L([nt])/L(n) \to 1$, as $n \to \infty$, for any $t > 0$; for definition and properties of regularly varying sequences, see [**9**]. Since for distributions having finite expectations the natural centering is with expectations, Mogyoródi's condition $EX_n = 0$ implied that centering constants a_n (from the formula (3)) were zero, and consequently the function A (from (4)) did not exist.

Let us compare Mogyoródi's conditions with our condition B. We shall see that if $EX_n = 0$, then the condition B is equivalent to the regular variation of the sequence b_n (from (3)). If $EX_n = 0$ and B is valid, we have:

$$\lim_{n\to\infty} P\{b_n^{-1} S_{[nt]} \le x\} = G_t(x), \qquad 0 < t \le 1.$$

Since then $a_n = 0$ we have

$$\lim_{n\to\infty} P\{b_n^{-1} S_n \le x\} = F(x),$$

and, using the convergence of types theorem ([**8**], [**9**]) it follows that there exists a non-negative function B such that (5) is valid:

$$\lim_{n\to\infty} \frac{b_{[nt]}}{b_n} = B(t), \qquad \text{for } 0 < t \le 1. \tag{5}$$

When $t > 1$ then it follows from (5) (since $b_{n+1}/b_n \to 1$, as $n \to \infty$ and $[nt] \le nt < [nt]+1$) that

$$\lim_{n\to\infty} \frac{b_{[nt]}}{b_n} = \lim_{n\to\infty} \frac{b_{[nt]}}{b_{[[nt]/t]}} = B^{-1}\left(\frac{1}{t}\right).$$

But that means exactly that the sequence b_n is regularly varying, so we have that $B(t) = t^\alpha$, $t > 0$. Since $b_n \to \infty$ as $n \to \infty$, then $\alpha > 0$. On the other hand, if Mogyoródi's conditions are satisfied, then for $0 < t \le 1$:

$$\begin{aligned} G_t(x) &= \lim_{n\to\infty} P\{b_n^{-1} S_{[nt]} \le x\} \\ &= \lim_{n\to\infty} P\{b_{[nt]}^{-1} S_{[nt]} \le b_n b_{[nt]}^{-1} x\} = F(t^{-\alpha} x). \end{aligned}$$

which means that B is valid.

Proof of Theorem 1. Put $p_{nj} = P\{\nu_n = j\}$, $A_n(x) = \sum_{j\le x} P_{nj}$. Since ν_n are independent of X_{nk}, $k \ge 1$, one has

$$P\left\{\sum_{k=1}^{\nu_n} X_{nk} \le x\right\} = \sum_{j=1}^{\infty} P\left\{\sum_{k=1}^{j} X_{nk} \le x\right\} p_{nj}.$$

In terms of characteristic functions (where f_{nk} is the characteristic function of X_{nk}) the right hand side of the preceding equality becomes

$$\int_0^\infty \prod_{k=1}^{[x]} f_{nk}(t)\, dA_n(x).$$

Put $x = yk_n$, $y > 0$; then the above integral becomes

$$\int_0^\infty \prod_{k=1}^{[yk_n]} f_{nk}(t)\, dA_n(yk_n). \tag{6}$$

Put $v_n(t) = \prod_{k=1}^{[yk_n]} f_{nk}(t)$, $y > 0$. We are interested whether the sequence of characteristic functions v_n tends to a limiting characteristic function as $n \to \infty$ and, if the answer is affirmative, to determine that limiting function.

From the condition A$'$ it follows, specially that if y is a positive integer, then

$$\lim_{n\to\infty} \prod_{k=1}^{[yk_n]} f_{nk}(t) = \lim_{n\to\infty} \prod_{k=1}^{yk_n} f_{nk}(t) = (\varphi(t))^y, \quad y \in \mathbf{N}.$$

Put $l_n = [yk_n]$. Since $k_n \to \infty$ as $n \to \infty$, obviously then

$$\lim_{n\to\infty} l_n/k_n = y. \tag{7}$$

Let us suppose first that $0 < y < 1$.

Denote by V_n the probability distribution corresponding to the characteristic function v_n. According to Helly's theorem we can select a convergent subsequence V_{n_j} converging to a nondecreasing and right continuous function V (in the sequel we shall put everywhere n instead of n_j in order to simplify the notation). In order that V be the probability distribution, it is necessary and sufficient that the sequence $\sum_{k=1}^{l_n} X_{nk}$ be *stochastically bounded*, i.e. that for every $\varepsilon > 0$ there exists b such that for big enough n

$$P\left\{ \left| \sum_{k=1}^{l_n} X_{nk} \right| > b \right\} < \varepsilon.$$

Let us denote by Z_{nk} random variables obtained from the variables X_{nk} by symmetrization. The condition A$'$ implies the convergence of

$$\sum_{k=1}^{k_n} Z_{nk} \tag{8}$$

as $n \to \infty$, which implies the stochastic boundedness of the sequence (8). From the symmetrization inequalities it follows that

$$P\left\{ \left| \sum_{k=1}^{k_n} Z_{nk} \right| > \varepsilon \right\} \geq \frac{1}{2} P\left\{ \left| \sum_{k=1}^{l_n} Z_{nk} \right| > \varepsilon \right\} \geq \frac{1}{4} P\left\{ \left| \sum_{k=1}^{l_n} X_{nk} - m_n \right| > \varepsilon \right\}. \tag{9}$$

$\varepsilon > 0$, where m_n is a median of the random variables

$$\sum_{k=1}^{l_n} X_{nk}.$$

The inequalities (9) imply that the sequence of random variables

$$\sum_{k=1}^{l_n} X_{nk} - m_n \tag{10}$$

is stochastically bounded.

From the continuity theorem for characteristic functions it follows that

$$\exp(-itm_n) \prod_{k=1}^{l_n} f_{nk}(t)$$

converges, as $n \to \infty$, to a characteristic function, which we shall denote by v. The characteristic function v is infinitely divisible, as limiting for the sums (10).

Put $u_n(t) = \prod_{k=k_n+l_n+1}^{2k_n} f_{nk}(t)$. According to the condition A$'$, we have that

$$\varphi(t) = \lim_{n\to\infty} v_n(t)u_n(t) = \lim_{n\to\infty} v_n(t)\exp(-itm_n)\exp(itm_n)u_n(t). \tag{11}$$

Since

$$\lim_{n\to\infty} v_n(t)\exp(-itm_n) = v(t), \tag{12}$$

it turns out that, as φ and v (being infinitely divisible) have no zeros,

$$\lim_{n\to\infty} \exp(itm_n)u_n(t) = u(t) \tag{13}$$

is uniquely determined and continuous at $t = 0$.

Let us consider the following k_n products (to the power $1/k_n$), each of them being obtained by multiplication of l_n the successive characteristic functions from the sequence $f_{n1}, \ldots, f_{nk_n}$ and multiplied by $\exp(-itm_n)$ (cyclic, f_{n1} follows after f_{nk_n}):

$$\begin{aligned}\Bigg(&\exp(-itm_n)\prod_{k=1}^{l_n} f_{nk}(t)\exp(-itm_n)\prod_{k=2}^{l_n+1} f_{nk}(t)\\ &\cdot\ldots\cdot\exp(-itm_n)f_{nk_n}(t)\prod_{k=1}^{l_n-1} f_{nk}(t)\Bigg)^{1/k_n}.\end{aligned} \tag{14}$$

Owing to the conditions A$'$, (12) and (13), each of the k_n products tends to the characteristic function v, so that the whole expression (14) tends to v, as $n \to \infty$. On the other hand, each function f_{nk}, $1 \le k \le k_n$, appears exactly l_n times in (14). Accordingly, (14) can be written as

$$\exp(-itm_n)\left(\prod_{k=1}^{k_n} f_{nk}(t)\right)^{l_n/k_n} \tag{15}$$

and we have that

$$\lim_{n\to\infty} \exp(-itm_n)\left(\prod_{k=1}^{k_n} f_{nk}(t)\right)^{l_n/k_n} = v(t).$$

By A$'$ and (7) we have

$$\lim_{n\to\infty} \left(\prod_{k=1}^{k_n} f_{nk}(t)\right)^{l_n/k_n} = (\varphi(t))^y.$$

Since v and φ are infinitely divisible characteristic functions, $\lim_{n\to\infty}\exp(-itm_n)$ exists and consequently $\lim_{n\to\infty} m_n$ is constant, which implies that $\sum_{k=1}^{l_n} X_{nk}$ is stochastically bounded, and therefore, without loss of generality, that constant could be taken to be zero. Therefore we have

$$v(t) = (\varphi(t))^y.$$

Up to now, everything holds just for the subsequence n_j of the sequence n. If we suppose that V_n contains another subsequence $V_{n'_j}$, which converges to a limit different from the one for the sequence n_j, then proceeding as above we get that every weakly convergent subsequence of V_n has the same limit.

So far it has been assumed that $0 < y < 1$. When $y > 1$ we have

$$\prod_{k=1}^{[yk_n]} f_{nk}(t) = \prod_{k=1}^{[y]k_n} f_{nk}(t) \prod_{k=[y]k_n+1}^{[yk_n]} f_{nk}(t). \tag{16}$$

From A$'$ it follows that $\prod_{k=1}^{[y]k_n} f_{nk}(t)$ tends to $\varphi^{[y]}$. From the case when $0 < y < 1$, analogously we have that

$$\lim_{n\to\infty} \prod_{k=[y]k_n+1}^{[yk_n]} f_{nk}(t) = (\varphi(t))^{y-[y]},$$

and, using (16), we get

$$\begin{aligned}
&\left| \int_0^\infty \prod_{k=1}^{[yk_n]} f_{nk}(t)\, dA_n(yk_n) - \int_0^\infty (\varphi(t))^y \, dA(y) \right| \\
&\le \left| \int_0^\infty \prod_{k=1}^{[yk_n]} f_{nk}(t)\, dA_n(yk_n) - \int_0^\infty (\varphi(t))^y \, dA_n(yk_n) \right| \\
&\quad + \left| \int_0^\infty (\varphi(t))^y \, dA_n(yk_n) - \int_0^\infty (\varphi(t))^y \, dA(y) \right|.
\end{aligned}$$

Denote by I_1 and I_2 the first and the second expression, respectively. Then I_2 tends to zero by extended Helly-Bray theorem, and for I_1 the following inequalities are valid:

$$\begin{aligned}
I_1 &\le \int_0^b \left| \prod_{k=1}^{[yk_n]} f_{nk}(t) - (\varphi(t))^y \right| dA_n(yk_n) \\
&\quad + \int_b^\infty \left| \prod_{k=1}^{[yk_n]} f_{nk}(t) - (\varphi(t))^y \right| dA_n(yk_n) \\
&\le \sup_{0\le y\le b} \left| \prod_{k=1}^{[yk_n]} f_{nk}(t) - (\varphi(t))^y \right| + 2(1 - A_n(b)).
\end{aligned}$$

Now, for each fixed $\varepsilon > 0$, we can select b such that $2(1-A(b)) < \varepsilon/2$ is valid. Since the convergence of the characteristic functions is uniform on finite intervals, there exists n_1 such that the first item is smaller then $\varepsilon/2$ for $n > n_1$. From the condition B it follows that n_2 exists such that $2(1 - A_n(b)) < \varepsilon/2$ for $n > n_2$. So, we have that $I_1 < \varepsilon$ for $n > \max\{n_1, n_2\}$, and the proof is completed.

Proof of Theorem 2. Set $p_{nk} = P\{\nu_n = k\}$ and

$$Q_n(x) = P\{\nu_n \le x\} = \sum_{k \le x} p_{nk}.$$

Since ν_n are independent of X_k, $n \ge 1$, $k \ge 1$, we have

$$P\left\{ b_n^{-1}\left(\sum_{j=1}^{\nu_n} X_j - a_n\right) \le x \right\} = \sum_{k=1}^{\infty} P\left\{ b_n^{-1}\left(\sum_{j=1}^{k} X_j - a_n\right) \le x \right\} p_{nk}.$$

Denote by h_n the characteristic function of $b_n^{-1}(S_{\nu_n} - a_n)$ and by f_j the characteristic function of X_j. Then, in terms of characteristic functions, the preceding equality becomes:

$$\begin{aligned} h_n(s) &= \sum_{k=1}^{\infty} \exp(-isa_n b_n^{-1}) \prod_{j=1}^{k} f_j(sb_n^{-1}) p_{nk} \\ &= \int_0^{\infty} \exp(-isa_n b_n^{-1}) \prod_{j=1}^{[x]} f_j(sb_n^{-1})\, dQ_n(x). \end{aligned}$$

Put $x = ny$; then

$$\begin{aligned} h_n(s) &= \int_0^{\infty} \exp(-isa_n b_n^{-1}) \prod_{j=1}^{[ny]} f_j(sb_n^{-1})\, dQ_n(ny) \\ &= \int_0^{\infty} \Big(\exp\big(is\big((a_{[ny]} - a_n)b_n^{-1} - a_{[ny]} b_n^{-1} b_{[ny]} b_{[ny]}^{-1}\big)\big) \times \\ &\quad \times \prod_{j=1}^{[ny]} f_j(b_{[ny]} b_n^{-1} s b_{[ny]}^{-1}) \Big)\, dQ_n(ny). \end{aligned}$$

By A and by (5) we have that

$$\lim_{n\to\infty} \exp(-isb_{[ny]} b_n^{-1} a_{[ny]} b_{[ny]}^{-1}) \prod_{j=1}^{[ny]} f_j(b_{[ny]} b_n^{-1} s b_{[ny]}^{-1}) = f(B(y)s).$$

Now let us consider $(a_{[ny]} - a_n)b_n^{-1}$. From the convergence of types theorem and from B we have that there exists a real function A such that:

$$\lim_{n\to\infty} (a_{[ny]} - a_n)b_n^{-1} = A(y), \qquad \text{for } 0 < y \le 1. \tag{17}$$

If $y > 1$, then (since for normed sums $(a_{n+1} - a_n)b_n^{-1} \to 0$, as $n \to \infty$):

$$\begin{aligned} \lim_{n\to\infty} (a_{[ny]} - a_n)b_n^{-1} &= \lim_{n\to\infty} -(a_{[[ny]/y]} - a_{[ny]}) b_{[ny]}^{-1} b_{[ny]} b_n^{-1} \\ &= -A(1/y)B(y). \end{aligned}$$

From (17) we have that

$$(a_{[nyz]} - a_n)b_n^{-1} = (a_{[nyz]} - a_{[ny]})b_{[ny]}b_{[ny]}^{-1}b_n^{-1} + (a_{[ny]} - a_n)b_n^{-1}$$

and therefore

$$A(yz) = A(z)B(y) + A(y) = A(y)B(z) + A(z).$$

so that ($z \neq 1$, $y \neq 1$)

$$A(z)(1 - B(z))^{-1} = A(y)(1 - B(y))^{-1}.$$

i.e. the function $A(\,\cdot\,)(1 - B(\,\cdot\,))^{-1}$ is equal to a constant which we shall denote by C. Therefore, for $y \neq 1$,

$$A(y) = A(z)(1 - B(z))^{-1}(1 - B(y)) = C(1 - y^\alpha) \tag{18}$$

is valid. Finally we obtain, using (18) and an argument similar to that from the last part of the proof of Theorem 1, that

$$\begin{aligned}\lim_{n\to\infty} h_n(s) &= \int_0^\infty \exp(isA(y))f(B(y)s)\,dQ(y)\\ &= \int_0^\infty \exp\bigl(isC(1 - y^\alpha)\bigr)f(sy^\alpha)\,dQ(y).\end{aligned}$$

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