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A COMMON FIXED POINT THEOREM OF WEAKLY COMMUTING MAPPINGS

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Abstract. By using a weak commutativity condition due to Sessa [**2**], we establish a common fixed point theorem for four selfmappings of a complete metric space. This result generalizes Theorem 1 of [**5**].

Throughout this paper (X, d) denotes a complete metric space, $\mathbf{R}^+$ the non-negative reals, and the function $\phi : [0,\infty)^5 \to [0,\infty)$ satisfies the following conditions:

(ϕ_1) ϕ is nondecreasing and upper semicontinuous in each coordinate variable;

(ϕ_2) For each $t > 0$

$$\varphi(t) = \max\big\{\phi(t,0,0,t,t), \phi(t,t,t,2t,0), \phi(t,t,t,0,2t)\big\} < t.$$

Let T, J be two self mappings of X. Sessa [**2**] defines T and J to be weakly commuting if $d(TJx, JTx) \le d(Tx, Jx)$ for all x in X. Two commuting mappings of X of course weakly commute but two weakly commuting mappings do not necessarily commute [**3**, Ex. 1].

By Theorem 1 of [**3**], we suppose that X contains at least three points.

Some fixed points theorems for weakly commuting mappings are proved in [**1**–**3**].

In a recent paper [**5**] the following theorem is proved

THEOREM 1. *Let f and g be continuous self-mappings of a complete metric space (X, d), $T, S : X \to X$ such that $T(X) \subset g(X)$, $S(X) \subset f(X)$, $Tf = fT$ and $Sg = gS$. If for all $x, y \in X$*

$$d(Tx, Sy) \le \phi\big(d(fx, gy), d(fx, Tx), d(gy, Sy), d(fx, Sy), d(gy, Ty)\big)$$

then each of the two pairs (T, f) and (S, g) has a unique common fixed point and these two points coincide.

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The purpose of this note is to prove a new common fixed point theorem for weakly commuting mappings in a complete metric space which improves and extends Theorem 1 for weakly commuting mappings.

THEOREM 2. *Let S, T, I, J be four self-mapings of X such that*: (1) $T(X) \subset I(X)$ *and* $S(X) \subset J(X)$,

(2) $d(Sx,Ty) \leq \phi\big(d(Ix,Jy), d(Ix,Sx), d(Jy,Ty), d(Ix,Ty), d(Jy,Sx)\big)$ *for all* $x, y \in X$.

If one *of S, T, I and J is continuous and S and T weakly commute respectively with I and J, then S, T, I, J have a common fixed point z. Furthermore, z is the unique common fixed point of S and I and of T and J.*

Proof. Let x_0 be an arbitrary point in X. Then, since (1) holds, we can define a sequence

$$\big\{Sx_0, Tx_1, Sx_2, \ldots, Sx_{2n}, Tx_{2n+1}, \ldots\big\} \tag{3}$$

inductively by $Sx_{2n} = Jx_{2n+1}$, $Tx_{2n+1} = Ix_{2n+2}$ for $n = 0, 1, 2, \ldots$. As in Theorem 1 of [**5**] the sequence (3) is a Cauchy sequence. By the completeness of X the sequence (3) converges to a point z in X, which is also the limit of the subsequence of (3) given by $\big\{Sx_{2n}\big\} = \big\{Jx_{2n+1}\big\}$ and $\big\{Tx_{2n-1}\big\} = \big\{Ix_{2n}\big\}$.

Let us first of all suppose that I is continuous. Then the sequences $\big\{ISx_{2n}\big\}$ and $\big\{I^2x_{2n}\big\}$ converge to Iz. Since S weakly commutes with I, we have

$$d(SIx_{2n}, Iz) \leq d(SIx_{2n}, ISx_{2n}) + d(ISx_{2n}, Iz) \leq d(Ix_{2n}, Sx_{2n}) + d(ISx_{2n}, Iz)$$

and on letting n tend to infinity it follows that the sequence $\big\{SIx_{2n}\big\}$ converges to Iz. Since S weakly commutes with I and ϕ is nondecreasing, using (2) we have

$$\begin{aligned} d(SIx_{2n}, Tx_{2n+1}) &\leq \phi\big(d(I^2x_{2n}, Jx_{2n+1}), d(I^2x_{2n}, SIx_{2n}), d(Jx_{2n+1}, Tx_{2n+1}), \\ &\qquad d(I^2x_{2n}, STIx_{2n+1}), d(Jx_{2n+1}, SIx_{2n})\big) \\ &\leq \phi\big(d(I^2x_{2n}, Jx_{2n+1}), d(I^2x_{2n}, SIx_{2n}), d(Jx_{2n+1}, Tx_{2n+1}), \\ &\qquad d(I^2x_{2n}, Tx_{2n+1}), d(Jx_{2n+1}, ISx_{2n}) + d(ISx_{2n}, SIx_{2n})\big) \\ &\leq \phi\big(d(I^2x_{2n}, Jx_{2n+1}), d(I^2x_{2n}, SIx_{2n}), d(Jx_{2n+1}, Tx_{2n+1}), \\ &\qquad d(I^2x_{2n}, Tx_{2n+1}), d(Jx_{2n+1}, ISx_{2n}) + d(Sx_{2n}, Ix_{2n})\big). \end{aligned}$$

Letting n tend to infinity and invoking the upper semicontinuity of ϕ we have

$$d(Iz, z) \leq \phi\big(d(Iz, z), 0, 0, d(Iz, z), d(Iz, z)\big) \leq \varphi(d(Iz, z)).$$

By Lemma 2 of [**4**] we have $Iz = z$. Again using (2) we have

$$\begin{aligned} d(Sz, Tx_{2n+1}) \leq \phi\big(&d(Iz, Jx_{2n+1}), d(Iz, Sz), d(Jx_{2n+1}, Tx_{2n+1}), \\ &d(Iz, Tx_{2n+1}), d(Jx_{2n+1}, Sz)\big) \end{aligned}$$

from which it follows, on letting n tend to infinity, that

$$d(Sz, z) \leq \phi\big(0, d(z, Sz), 0, 0, d(z, Sz)\big) < \varphi(d(Sz, z)).$$

By Lemma 2 of [**4**] we have $Sz = z$. Since Sz is in $SX \subset JX$, there exists a point z' in X such that $Jz' = z$. By (2) we have

$$\begin{aligned} d(z, Tz') = d(Sz, Tz') &\le \phi\big(d(Iz, Jz'), d(Iz, Sz), d(Jz', Tz'), d(Iz, Tz'), d(Jz', Sz)\big) \\ &= \phi\big(0, 0, d(z, Tz'), d(z, Tz'), 0\big) < \varphi(d(Iz, Tz')). \end{aligned}$$

By Lemma 2 of [**4**] we have $Tz' = z$. As T and J weakly commute we have

$$d(Tz, Jz) = d(TJz', JTz') < d(Jz', Tz') = d(z, z') = 0$$

giving $Tz = TJz' = JTz' = Jz$. Thus from (2)

$$\begin{aligned} d(z, Tz) = d(Sz, Tz) &\le \phi\big(d(Iz, Jz), d(Iz, Sz), d(Jz, Tz), d(Iz, Tz), d(Jz, Sz)\big) \\ &= \phi\big(d(z, Tz), 0, 0, d(z, Tz), d(z, Tz)\big) < \varphi(d(z, Tz)) \end{aligned}$$

and so $Tz = z$ by Lemma 2 of [**4**].

The same result of course holds if we suppose that J is continuous instead of I.

Now let us suppose that the mapping S is continuous, so that $\{S^2x_{2n}\}$ and $\{SIx_{2n}\}$ converge to Sz. Since S and I weakly commute, it follows as above that the sequence $\{ISx_{2n}\}$ converges to the point Sz. Since S weakly commutes with I and ϕ is nondecreasing, using (2) we have

$$\begin{aligned} d(S^2x_{2n}, Tx_{2n+1}) \le \phi\big(&d(ISx_{2n}, Jx_{2n+1}), d(ISx_{2n}, S^2x_{2n}), d(Jx_{2n+1}, Tx_{2n+1}), \\ &d(ISx_{2n}, Tx_{2n+1}), d(Jx_{2n+1}, S^2x_{2n})\big). \end{aligned}$$

Letting n tend to infinity, we have

$$d(Sz, z) \le \phi\big(d(Sz, z), 0, 0, d(Sz, z), d(Sz, z)\big) < \varphi(d(Sz, z)).$$

By Lemma 2 of [**4**] we have $Sz = z$.

Once again there exists a point z' in X such that $Jz' = z$. Thus

$$\begin{aligned} d(S^2x_{2n}, Tz') \le \phi\big(&d(ISx_{2n}, Jz'), d(ISx_{2n}, S^2x_{2n}), d(Jz', Tz'), \\ &d(ISx_{2n}, Tz'), d(Jz', S^2x_{2n})\big). \end{aligned}$$

Letting n tend to infinity, it follows that

$$d(z, Tz') \le \phi\big(0, 0, d(z, Tz'), d(z, Tz'), 0\big) < \varphi(d(z, Tz')).$$

By Lemma 2 of [**4**] we have $Tz' = z$. Since T and J weakly commute, it again follows as above that $Tz = Jz$. Furthermore,

$$d(Sx_{2n}, Tz) \le \phi\big(d(Ix_{2n}, Jz), d(Ix_{2n}, Sx_{2n}), d(Jz, Tz), d(Ix_{2n}, Tz), d(Jz, Sx_{2n})\big).$$

Letting n tend to infinity, it follows that

$$d(z, Tz) \le \phi\big(d(z, Tz), 0, 0, d(z, Tz), d(z, Tz)\big) < \varphi(d(z, Tz))$$

and so $Tz = z = Jz$ by Lemma 2 of [**4**]. The point z is therefore in the range of T and since the range of I contains the range of T, there exists z'' in X such that $Iz'' = z$. Thus

$$\begin{aligned} d(Sz'', z) &= d(Sz'', Tz) \\ &\leq \phi\big(d(Iz'', Jz), d(Iz'', Sz''), d(Jz, Tz), d(Iz'', Tz), d(Jz, Sz'')\big) \\ &= \phi\big(0, d(z, Sz''), 0, 0, d(z, Sz'')\big) < \varphi(d(z, Sz'')) \end{aligned}$$

and so $Sz'' = z$ by Lemma 2 of [**4**]. Since S and I weakly commute we have

$$d(Sz, Iz) = d(SIz'', ISz'') \leq d(Iz'', Sz'') = d(z, z) = 0.$$

Thus $Sz = Iz = z$. We have therefore proved once again that z is a common fixed point of S, T, I and J.

If the mapping T is continuous instead of S, then the proof that z is again a common fixed point of S, T, I and J is similar.

Now let w be a second common fixed point of S and I. Using inequality (2), we have

$$\begin{aligned} d(w, z) = d(Sw, Tz) &\leq \phi\big(d(Iw, Jz), d(Iw, Sw), d(Jw, Tz), d(Iw, Tz), d(Jz, Sw)\big) \\ &= \phi\big(d(w, z), 0, 0, d(w, z), d(w, z)\big) < \varphi(d(w, z)) \end{aligned}$$

and by Lemma 2 of [**4**] it follows that $w = z$. Then z is the unique common fixed point of S and I. Similarly it is proved that z is the unique common fixed point of T and J.

This completes the proof of the theorem.

Corollary. *Let S, T, I, J be self-mapings of a complete metric space (X, d) such that*

(4) $T(X) \subset I(X)$ *and* $S(X) \subset J(X)$.

(5) *There is a function ϕ sastisfying (ϕ_1) and*

(ϕ_2') $\phi(t, t, t, at, bt) < t$ *for all $t > 0$, where $a + b \leq 2$ and for all $x, y \in X$ the inequality* (2) *holds.*

If one of S, T, I and J is continuous and if S and T weakly commute respectively with I and J, then S, T, I and J have a common fixed point z. Furthermore z is the unique common fixed point of S and I and of T and J.

Corollary 2. *Let f, g, T and S be self mappings of a complete metric space (X, d). f and g are continuous. Suppose $Tf = fT$, and $Sg = gS$, $TX \subset gX$ and $SX \subset fX$. If there is a function ϕ satisfying (ϕ_1) and (ϕ_2') where $a + b = 2$ and if for all $x, y \in X$ the inequality* (2) *holds, then each of the pairs (T, f) and (S, g) has a unique common fixed point and these two points coincide* [**5**].

The main theorem of [**6**] is a special case of Corollary 2.

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