PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série tome 47 (61), 1990, 127–131

CONCERNING SPLITTABILITY AND PERFECT MAPPINGS

A. V. Arhangel'skiĭ and **Lj. D. Kočinac**

Abstract. We consider the following question: let a space X admits a perfect mapping onto a space Y from some class $\mathcal{P}$ of topological spaces and let X be splittable over $\mathcal{P}$. Does X belong to $\mathcal{P}$?

0. Introduction

The notions of splittability, $\mathcal{P}$-splittability and $(\mathcal{M},\mathcal{P})$-splittability introduced recently by A. V. Arhangel'skiĭ have been the subject of several papers: [**4**], [**6**], [**11**], [**12**]. The definitions are as follows:

Let $\mathcal{M}$ be a class of continuous mappings and $\mathcal{P}$ a class of topological spaces. A space X is called $(\mathcal{M},\mathcal{P})$-*splittable* or $\mathcal{M}$-*splittable over* $\mathcal{P}$ if for every $A \subset X$ there exist some $Y \in \mathcal{P}$ and a mapping $f \in \mathcal{M}$ from X onto Y such that $f^{-1}f(A) = A$. When $\mathcal{M}$ is the class of all continuous (perfect) mappings we use the term *splittable over* $\mathcal{P}$ (*perfectly splittable over* $\mathcal{P}$) instead of $(\mathcal{M},\mathcal{P})$-splittable (see [**4**]).

Clearly, if there is a continuous bijection from a space X onto a space $Y \in \mathcal{P}$, then X is splittable over $\mathcal{P}$ and we can say that X is absolutely splittable over $\mathcal{P}$ in this case. So splittability is a generalization of continuous bijections.

A number of theorems in general topology can be formulated in the following form: Let $\mathcal{P}$ be a topological property. If a space X admits a perfect mapping onto a space Y satisfying $\mathcal{P}$ and a one-to-one mapping onto a space Z satisfying $\mathcal{P}$, then X satisfies $\mathcal{P}$ (see, for example, [**5**], [**9**]). This suggests the following natural question which is the subject of this article: when splittability over $\mathcal{P}$ replaces one-to-one mappings in such theorems; more precisely: let a space X admit a perfect mapping onto a space from a class $\mathcal{P}$ and let X be splittable over a class Q. Does X belong to $\mathcal{P}$ or $\mathcal{Q}$?

AMS Subject Classification (1980): Primary 54 C 10, 54 D 30, 54 D 55, 54 E 18, 54 E 30

The second author was supported by RZN Srbije.

This paper was written while the first author was visiting the University of Niš in April, 1989.

All spaces in this article are T_2 (unless stated otherwise) and all mappings are continuous and onto. Recall that a mapping $f : X \to Y$ is perfect if f is closed with $f^{-1}(y)$ compact for each $y \in Y$. We use the usual notation and terminology [**5**], [**9**] and give references (although not necessarily the original source) where the definitions of undefined concepts can be found.

Let us begin with the following simple but useful result [**11**]:

LEMMA 0.1. *Let $\mathcal{M}$ be the class of all closed mappings. If a space X is $\mathcal{M}$-splittable over the class of Hausdorff (regular, Tychonoff, normal) spaces then X in Hausdorff (regular, Tychonoff, normal).*

In the sequel we shall use the following well known result.

LEMMA 0.2. *If $f : X \to Y$ is a perfect mapping then for any subset $B \subset Y$ the restriction $f_B : f^{-1}(B) \to B$ is perfect.*

We shall often use

LEMMA 0.3. *Let $\mathcal{P}$ be a class of topological spaces which is hereditary and finitely multiplicative. Suppose that a space X is splittable over $\mathcal{P}$ and admits a perfect mapping onto a space from $\mathcal{P}$. Then*

(i) *X is perfectly splittable over $\mathcal{P}$;*

(ii) *every $A \subset X$ admits a perfect mapping onto a space from $\mathcal{P}$.*

Proof. (i) Let $f : X \to Y \in \mathcal{P}$ be perfect. For every $A \subset X$ there are a space $Z \in \mathcal{P}$ and a mapping $g : X \to Z$ such that $g^{-1}g(A) = A$. Since f is a perfect mapping the diagonal product $\varphi = f\Delta g : X \to Y \times Z$ is also perfect — that is well known. Moreover, $\varphi^{-1}\varphi(A) = A$. This means that X is perfectly splittable over the class $\mathcal{P}$ as $Y \times Z \in \mathcal{P}$.

(ii) This follows from the fact that $\varphi(A) \in \mathcal{P}$ (because $\mathcal{P}$ is hereditary) together with Lemma 0.2.

1. Moore spaces and σ-spaces

Let us recall some definitions. A network for a space X is a collection $\mathcal{N}$ of subsets of X such that for every $x \in X$ and every open set U with $x \in U$ there is an $A \in \mathcal{N}$ such that $x \in A \subset U$. The net weight $\mathrm{nw}(X)$ of a space X is the least cardinality of a network for X. A cosmic space is a regular space with a countable network. A space X is a σ-space if it has a σ-discrete network. The definition of Moore spaces can be found in [**10**], for example.

THEOREM 1.1. *If a space X is splittable over the class $\mathcal{P}$ of spaces of weight (net weight) $\leq \tau$ and admits a perfect mapping onto a space of weight (net weight) $\leq \tau$, then X has weight (net weight) $\leq \tau$.*

Proof. Let us note that the class $\mathcal{P}$ is hereditary and finitely multiplicative so that we can apply Lemma 0.3. Hence every subspace $A \subset X$ can be mapped

by a perfect mapping onto a space of weight (net weight) $\leq \tau$. Then the theorem follows from the following result of Arhangel'skiĭ-Pytkeev (see [**3**] and [**14**]): if X is a Hausdorff space and every subspace of X admits a perfect mapping onto a space of weight (net weight) $\leq \tau$, then X itself has weight (net weight) $\leq \tau$.

COROLLARY 1.2. *If a Lindelöf p-space X* [**5**] *is splittable over the class of spaces of countable weight, then X has a countable base.*

A similar result can be formulated for perfectly Lindelöf spaces (=spaces which admit a perfect mapping onto a space with a countable network).

From Lemma 0.1 and Theorem 1.1 we get the following

COROLLARY 1.3. *If a space X is splittable over the class of cosmic spaces and admits a perfect mapping onto a cosmic space, then X is cosmic.*

Remark 1.4. Following [**4**] (see also [**11**]) denote by $\mathrm{w}_{\mathrm{ps}}(X)$, X is a space, $\min\{\tau : X$ is perfectly splittable over the class of all spaces Y with $\mathrm{w}(Y) \leq \tau\}$. Then Theorem 1.1 gives us a method to prove: $\mathrm{w}(X) = \mathrm{w}_{\mathrm{ps}}(X)$; this was proved in [**11**] by a different manner. Similarly we have $\mathrm{nw}(X) = \mathrm{nw}_{\mathrm{ps}}(X)$ for every space X (see also [**12**]).

THEOREM 1.5. *If a space X is splittable over the class $\mathcal{P}$ of σ-spaces and admits a perfect mapping onto a σ-space, then X is a σ-space.*

Proof. Let $f : X \to Y \in \mathcal{P}$ be a perfect mapping and let Y be an arbitrary subset of X. Then there exist a space $Z \in \mathcal{P}$ and a mapping $g : X \to Z$ such that $g^{-1}g(A) = A$. Put $\varphi = f\Delta g$. As in Lemma 0.3 φ is perfect and $\varphi^{-1}\varphi(A) = A$. The set $\varphi(A)$ is a σ-space because it is a subspace of $Y \times Z$ which is a σ-space. Thus $\varphi(A)$ is a strong Σ-space (see [**10**])) and consequently A is also a strong Σ-space as this property is an inverse invariant under perfect mappings. Therefore X is a hereditarily strong Σ-space. On the other hand, X is a perfect space. Indeed, if A is closed in X then $\varphi(A)$ is closed in (a perfect) space $Y \times Z$ so that $\varphi(A)$ is a G_δ-set. But then $A = \varphi^{-1}\varphi(A)$ is a G_δ-set in X. Now we have to apply the following result of Z. Balogh [**8**]: a perfect space X is a σ-space if and only if it is a hereditarily strong Σ-space. The theorem is proved.

Now we are going to prove that a similar result is true for the class of Moore spaces (this class is a subclass of the class of σ-spaces).

THEOREM 1.6. *If a Tychonoff space X admits a perfect mapping f onto a Moore space Y and is splittable over the class of Moore spaces, then X is also a Moore space.*

Proof. Using notation from Lemma 0.3 and Theorem 1.5 we have that $\varphi(A)$ is a Moore space. It is known that the perfect inverse image (with completely regular domain) of a Moore space is a subparacompact [**5**], [**10**] p-space [**5**], [**10**]. So, every $A \subset X$ is a p-space, i.e. X is hereditarily p-space. Also, every $A \subset X$

is θ-refinable [**10**], because every subparacompact space is θ-refinable. The space X is perfect — that can be shown as in Theorem 1.5. In [**14**] Pytkeev has proved that hereditarily p-spaces are developable if and only if they are perfect. Hence X is a developable space and as X is (completely) regular, X is a Moore space. The theorem is proved.

It should be noted that in [**6**] it was proved (using a result of Balogh-Pytkeev [**8**], [**14**]) that every paracompact p-space splittable over the class of metrizable spaces is also metrizable.

2. Convergence properties

All undefined concepts can be found in [**1**], [[**2**], [**13**].

THEOREM 2.1. *If a space X admits a perfect mapping f onto a $\langle 3\text{-}FU\rangle$-space Y and is splittable over the class of countably compact FU-spaces, then X is a Fréchet-Urysohn space.*

Proof. Note that X is a k-space and prove first that $t(X) \leq \aleph_0$. According to a result of Rančin [**15**] for this it is enough to prove that for every compact $B \subset X$ one has $t(B) \leq \aleph_0$. We have that B is splittable over the class of spaces of countable tightness. As B is compact the tightness of B is also countable as was proved by Arhangel'skiĭ in [**4**].

Now, let T be any countable subset of X. Fix a continuous mapping $g : X \to Z$ onto some countably compact FU space Z such that $g^{-1}g(T) = T$. Consider $\varphi = f\Delta g$. The space $Y \times Z$ is an FU-space (see [**2**]) so that $\varphi(T)$ is also an FU-space and thus it is a k-space. As the property being a k-space is an inverse invariant under perfect mappings we have that T is a k-space (according to Lemma 0.2). Since every subspace of T is countable we have: every subspace of T is a k-space, so that T is an FU-space by the well known result of Arhangel'skiĭ. Hence we get: $t(X) \leq \aleph_0$ and every countable subset of X is an FU-space. From this it follows that X is a Fréchet-Urysohn space (see [**2**], [**13**]). The theorem is proved.

THEOREM 2.2. *If a space X admits a perfect mapping f onto a bisequential space Y and is splittable over the class of all bisequential spaces, then X is an $\aleph_0$-bisequential space.*

Proof. The proof is similar to the proof of Theorem 2.1. We use the same notation as in that theorem. The space $Y \times Z$ is bisequential, so that $\varphi(A)$ is bisequential. Thus $\varphi(A)$ is a bi-k-space. The property being a bi-k-space is an inverse invariant under perfect mappings so that A is bi-k in X. Hence every subspace of X is a bi-k-space. From a result of Arhangel'skiĭ [**1**] it follows that X is an $\aleph_0$-bisequential space.

Remark 2.3. In a similar way it can be proved: if a space X admits a perfect mapping onto an $\aleph_0$-bisequential space and is splittable over the class of strongly FU-spaces, then X is FU.

REFERENCES

[1] А. В. Архангельский, *Спектр частот топологического пространства и классификация пространств*, ДАН СССР **206** (1972), 255–258.

[2] А. В. Архангельский, *Спектр частот топологического пространства и операция произведения*, Труды Моск. матем. об-ва **40** (1979), 171–206.

[3] A. V. Arhangel'skiĭ, *On hereditary properties*, Gen. Topol. Appl. **3** (1973), 39–46.

[4] А. В. Архангельский, *Некоторые новые направления в теории непрерывных отображений*, В: *Непрерывные функции на топологических пространствах*, ЛГУ, Рига, 1986, 5–35.

[5] А. В. Архангельский, В. И. Пономарев, *Основы общей топологии в задачах и упражне-
ниях*, Наука, Москва, 1974.

[6] А. В. Архангельский, Д. Б. Шахматов, *О поточечной аппроксимации произвольных функций счетными семействами непрерывных функций*, Труды семинара им. И. Г. Петровского **13** (1988), 206–227.

[7] Z. Balogh, *On the metrizability of F_{pp}-spaces and its relationship to the normal Moore space conjecture*, Fund. Math. **113** (1981), 45–58.

[8] Z. Balogh, *On hereditarily strong Σ-spaces*, Topology Appl. **17** (1984), 199–215.

[9] R. Engelking, *General Topology*, PWN, Warszawa, 1977.

[10] G. Gruenhage, *Generalized metric spaces*, In; K. Kunen and J. E. Vaughan (eds.), *Handbook of Set-theoretic Topology*, North Holland, Amsterdam, 1984, 423–501.

[11] Lj. D. Kočinac, *On (M, P)-splittability of topological spaces*, (Proc. IV Meeting on Topology, Sorrento 1988.), Rend. Circ. Mat. Palermo (1990), to appear.

[12] Lj. Kočinac, *Perfect $\mathcal{P}$-splittability of topological spaces*, Zbornik Rad. Fil. Fak. (Niš), Ser. Mat. **3** (1989), 19–24.

[13] E. Michael, *A quintuple quotient quest*, Gen. Topol. Appl. **2** (1972), 91–138.

[14] Е. Г. Пыткеев, *О наследственно перистых пространствах*, Мат. заметки **28** (1980), 603–618.

[15] Д. В. Ранчин, *Теснота, секвенциальность и замкнутые покрытия*, ДАН СССР **232** (1977), 1015–1018.

Chair of General Topology and Geometry,
Department of Mathematics,
Moscow State University,
Moscow, USSR

Filozofski fakultet
18000 Niš, p.p. 91
Yugoslavia

(Received 19 05 1989)