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ON SOME HYPERSURFACES OF A RECURRENT RIEMANNIAN SPACE

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Abstract. We consider quasiumbilical hypersurface of a recurrent Riemannian space and find the conditions for such hypersurface to be conformally recurrent. We prove (assuming some conditions) that the curvatutre tensor of such hypersurface has the form (3.10) and discuss the case when the hypersurface is conformally flat.

1. Introduction. Let $(\overline{M}, \overline{g})$ be an $(n+1)$-dimensional $(n>3)$ Riemannian space covered by a system of coordinate neighbourhoods (U, y^α). Let (M, g) be a hypersurface of (M, g) defined in a local coordinate system by means of the system of parametric equations $y^\alpha = y^\alpha(x^i)$, where g is the induced metric. Here and in the sequel, Greek indices take the values $1, 2, \dots, n+1$, and Latin indices — the values $1, 2, \dots, n$. Let N^α be a local unit normal to (M, g) and let $B_i^\alpha = \partial y^\alpha / \partial x^i$. Then

$$g_{ij} = \overline{g}_{\alpha\beta} B_i^\alpha B_j^\beta, \tag{1.1}$$

$$\overline{g}_{\alpha\beta} N^\alpha B_i^\beta = 0, \qquad \overline{g}_{\alpha\beta} N^\alpha N^\beta = \varepsilon, \quad \varepsilon = \pm 1, \tag{1.2}$$

$$B_i^\alpha B_j^\beta g^{ij} = \overline{g}^{\alpha\beta} - \varepsilon N^\alpha N^\beta. \tag{1.3}$$

We denote by $\overline{R}_{\alpha\beta\gamma\delta}$, $\overline{R}_{\alpha\beta}$ and $\overline{R}$ the curvature tensor, the Ricci tensor and the scalar curvature of $(\overline{M}, \overline{g})$ respectively and by R_{ijkl}, R_{ij} and R the corresponding objects of the hypersurface. Let h be the second fundamental form of the hypersurface and let ∇ be the operator of the van der Waerden-Bortolotti covariant derivative. Then the Gauss and Codazzi equations for (M, g) of $(\overline{M}, \overline{g})$ can be written in the form (c.f.e.g. [**1**], p. 149)

$$\overline{R}_{\alpha\beta\gamma\delta} B_i^\alpha B_j^\beta B_k^\gamma B_l^\delta = R_{ijkl} - \varepsilon(h_{il} h_{jk} - h_{ik} h_{jl}),$$
$$\overline{R}_{\alpha\beta\gamma\delta} N^\alpha B_j^\beta B_k^\gamma B_l^\delta = \nabla_l h_{jk} - \nabla_k h_{jl}.$$

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Also (c.f. [**1**, pp. 147–148])

$$\nabla_r B_j^\beta = \varepsilon h_{rj} N^\beta, \qquad \nabla_r N^\alpha = -h_{ra} g^{at} B_t^\alpha. \tag{1.4}$$

If there exist on (M,g) two functions α, β and a 1-form v such that

$$h_{ij} = \alpha g_{ij} + \beta v_i v_j, \tag{1.5}$$

(M,g) is said to be quasiumbilical ([**2**, p. 147]). If $\beta = 0$, (M,g) is an umbilical hypersurface. Miyazawa and Chūman [**3**] investigated totally umbilical subspaces of recurrent Riemannian space. Among others, they proved that such subspece is conformally recurrent (c.f. also [**4**]).

In this paper we consider quasiumbilical hypersurface of a recurrent Riemannian space assuming

$$\nabla_k v_i = \sigma g_{ik} + t_k v_i, \tag{1.6}$$

where t_k and σ are a covariant vector field and a scalar function on M respectively. Also, we suppose $v_a v^a \neq 0$.

The aim of this paper is to find the necessary and sufficient conditions for such hypersurface to be conformally recurrent. After that, we prove that if $\alpha \neq 0$, $\alpha n + \beta v_a v^a \neq 0$, the curvature tensor of the hypersurface has the form (3.10) and discuss the case when the hypersurface is conformally flat.

Using (1.5) and (1.6), we can rewrite the Gauss and Codazzi equations as follows

$$\begin{aligned}\overline{R}_{\alpha\beta\gamma\delta} B_i^\alpha B_j^\beta B_k^\gamma B_l^\delta &= R_{ijkl} - \varepsilon\alpha^2 (g_{il} g_{jk} - g_{ik} g_{jl}) \\ &\quad - \varepsilon\alpha\beta (g_{il} v_j v_k + g_{jk} v_i v_l - g_{ik} v_j v_l - g_{jl} v_i v_k),\end{aligned} \tag{1.7}$$

$$\overline{R}_{\alpha\beta\gamma\delta} N^\alpha B_j^\beta B_k^\gamma B_l^\delta = g_{jk} w_l - g_{jl} w_k + v_j (v_k T_l - v_l T_k), \tag{1.8}$$

where

$$w_l = \alpha_l - \beta\sigma v_l, \quad T_l = \beta_l + 2\beta t_l, \quad \alpha_l = \frac{\partial\alpha}{\partial x^l}, \quad \beta_l = \frac{\partial\beta}{\partial x^l}. \tag{1.9}$$

As for (1.4), they became

$$\nabla_r B_j^\beta = \varepsilon(\alpha g_{rj} + \beta v_r v_j) N^\beta, \tag{1.10}$$

$$\nabla_r N^\alpha = -(\alpha B_r^\alpha + \beta v_r v^t B_t^\alpha). \tag{1.11}$$

From (1.6), we obtain

$$\nabla_l \nabla_k v_j - \nabla_k \nabla_l v_j = g_{kj}(\sigma_l - \sigma t_l) - g_{lj}(\sigma_k - \sigma t_k) + v_j(\nabla_l t_k - \nabla_k t_l).$$

Using the Ricci identity, we get

$$-v_a R^a{}_{jkl} = g_{jk} S_l - g_{lj} S_k + v_j M_{lk}, \tag{1.12}$$

where

$$\sigma_l = \frac{\partial\sigma}{\partial x^l}, \qquad S_l = \sigma_l - \sigma t_l, \qquad M_{lk} = \nabla_l t_k - \nabla_k t_l. \tag{1.13}$$

2. Quasiumbilical hypersurface of a recurrent Riemannian space. Applying the operator ∇_r to (1.7) and using (1.10), we obtain

$$\begin{aligned}
&\nabla_\rho \overline{R}_{\alpha\beta\gamma\delta} B^\rho_r B^\alpha_i B^\beta_j B^\gamma_k B^\delta_l \\
&\quad + \varepsilon(\alpha g_{ri} + \beta v_r v_i)\overline{R}_{\alpha\beta\gamma\delta} N^\alpha B^\beta_j B^\gamma_k B^\delta_l - \varepsilon(\alpha g_{rj} + \beta v_r v_j)\overline{R}_{\beta\alpha\gamma\delta} N^\beta B^\alpha_i B^\gamma_k B^\delta_l \\
&\quad + \varepsilon(\alpha g_{rk} + \beta v_r v_k)\overline{R}_{\gamma\delta\alpha\beta} N^\gamma B^\delta_l B^\alpha_i B^\beta_j - \varepsilon(\alpha g_{rl} + \beta v_r v_l)\overline{R}_{\delta\gamma\alpha\beta} N^\delta B^\gamma_k B^\alpha_i B^\beta_j \\
&= \nabla_r \big[R_{ijkl} - \varepsilon\alpha^2(g_{il}g_{jk} - g_{ik}g_{jl}) \\
&\quad -\varepsilon\alpha\beta(g_{il}v_j v_k + g_{jk}v_i v_l - g_{ik}v_j v_l - g_{jl}v_i v_k)\big].
\end{aligned}$$

Substituting (1.8) into this equation, we find

$$\begin{aligned}
&\nabla_\rho \overline{R}_{\alpha\beta\gamma\delta} B^\rho_r B^\alpha_i B^\beta_j B^\gamma_k B^\delta_l \\
&\quad + \varepsilon\alpha\big[(g_{ri}g_{jk} - g_{rj}g_{ik})w_l + (g_{rj}g_{il} - g_{ri}g_{jl})w_k \\
&\qquad + (g_{rk}g_{li} - g_{rl}g_{ik})w_j + (g_{rl}g_{kj} - g_{rk}g_{lj})w_i\big] \\
&\quad + \varepsilon\beta v_r\big[g_{jk}(v_i w_l + v_l w_i) + g_{il}(v_k w_j + v_j w_k) \\
&\qquad - g_{jl}(v_i w_k + v_k w_i) - g_{ik}(v_j w_l + v_l w_j)\big] \\
&\quad + \varepsilon\alpha\big[g_{ri}v_j(v_k T_l - v_l T_k) + g_{rj}v_i(v_l T_k - v_k T_l) \\
&\qquad + g_{rk}v_l(v_i T_j - v_j T_i) + g_{rl}v_k(v_j T_i - v_i T_j)\big] \\
&= \nabla_r\big[R_{ijkl} - \varepsilon\alpha^2(g_{il}g_{jk} - g_{ik}g_{jl}) \\
&\qquad - \varepsilon\alpha\beta(g_{il}v_j v_k + g_{jk}v_i v_l - g_{ik}v_j v_l - g_{jl}v_i v_k)\big].
\end{aligned}$$

In view of (1.6), this can be rewritten in the form

$$\begin{aligned}
\nabla_r R_{ijkl} &= \nabla_\rho \overline{R}_{\alpha\beta\gamma\delta} B^\rho_r B^\alpha_i B^\beta_j B^\gamma_k B^\delta_l + 2\varepsilon\alpha\alpha_r(g_{il}g_{jk} - g_{ik}g_{jl}) \\
&\varepsilon\big[(\alpha\beta)_r + 2\alpha\beta t_r - 2\beta^2\sigma v_r\big](g_{il}v_j v_k + g_{jk}v_i v_l - g_{ik}v_j v_l - g_{jl}v_i v_k) \\
&\quad + \varepsilon\alpha\big[(g_{ri}g_{jk} - g_{rj}g_{ik})\alpha_l + (g_{rj}g_{il} - g_{ri}g_{jl})\alpha_k \\
&\qquad + (g_{rk}g_{li} - g_{rl}g_{ik})\alpha_j + (g_{rl}g_{kj} - g_{rk}g_{lj})\alpha_i\big] \\
&\quad + \varepsilon\beta v_r\big[g_{jk}(v_i\alpha_l + v_l\alpha_i) + g_{il}(v_k\alpha_j + v_j\alpha_k) \\
&\qquad - g_{jl}(v_i\alpha_k + v_k\alpha_i) - g_{ik}(v_j\alpha_l + v_l\alpha_j)\big] \\
&\quad + \varepsilon\alpha\big[g_{ri}v_j(v_k T_l - v_l T_k) + g_{rj}v_i(v_l T_k - v_k T_l) \\
&\qquad + g_{rk}v_l(v_i T_j - v_j T_i) + g_{rl}v_k(v_j T_i - v_i T_j)\big].
\end{aligned}$$

Now, let us suppose that the $(\overline{M}, \overline{g})$ is a recurrent Riemannian space, i.e.

$$\nabla_\rho \overline{R}_{\alpha\beta\gamma\delta} = a_\rho \overline{R}_{\alpha\beta\gamma\delta}. \tag{2.1}$$

Taking into account (1.7), the preceding relation becomes

$$\begin{aligned}
\nabla_r R_{ijkl} &= a_r R_{ijkl} + \varepsilon(2\alpha\alpha_r - \alpha^2 a_r)(g_{il}g_{jk} - g_{ik}g_{jl}) \\
&\quad + \varepsilon\big[(\alpha\beta)_r + 2\alpha\beta t_r - 2\beta^2\sigma v_r - \alpha\beta a_r\big](g_{il}v_j v_k + g_{jk}v_i v_l - g_{ik}v_j v_l - g_{jl}v_i v_k)
\end{aligned}$$

$$
\begin{aligned}
&+\varepsilon\alpha\big[(g_{ri}g_{jk}-g_{rj}g_{ik})\alpha_l+(g_{rj}g_{il}-g_{ri}g_{jl})\alpha_k\\
&\quad+(g_{rk}g_{il}-g_{rl}g_{ik})\alpha_j+(g_{rl}g_{kj}-g_{rk}g_{lj}\alpha_i)\big]\\
&+\varepsilon\beta v_r\big[g_{jk}(v_i\alpha_l+v_l\alpha_i)+g_{il}(v_k\alpha_j+v_j\alpha_k)\\
&\quad-g_{jl}(v_i\alpha_k+v_k\alpha_i)-g_{ik}(v_j\alpha_l+v_l\alpha_j)\big]\\
&+\varepsilon\alpha\big[g_{ri}v_j(v_kT_l-v_lT_k)+g_{rj}v_i(v_lT_k-v_kT_l)\\
&\quad+g_{rk}v_l(v_iT_j-v_jT_i)+g_{rl}v_k(v_jT_i-v_iT_j)\big],
\end{aligned}
\tag{2.2}
$$

where

$$a_r=a_\rho B^\rho{}_r.$$

From (2.2), we have

$$
\begin{aligned}
\nabla_rR_{jk}&=a_rR_{jk}\\
&+\varepsilon g_{jk}\big\{(2n\alpha+\beta v_av^a)\alpha_r+\alpha v_av^a\beta_r-\alpha[(n-1)\alpha+\beta v_av^a]a_r\\
&\quad+2\alpha\beta v_av^at_r+2\beta[\alpha_av^a-\beta\sigma v_av^a]v_r\big\}\\
&+\varepsilon v_jv_k\big[(n-2)\beta a_r+n\alpha\beta_r+2n\alpha\beta t_r-2(n-2)\beta^2\sigma v_r\\
&\quad-(n-2)\alpha\beta a_r\big]\\
&+\varepsilon\alpha g_{rj}\big[(n-2)\alpha_k+v_av^aT_k-T_av^av_k\big]\\
&+\varepsilon\alpha g_{rk}\big[(n-2)\alpha_j+v_av^aT_j-T_av^av_j\big]\\
&+\varepsilon v_rv_k\big[(n-2)\beta\alpha_j-\alpha T_j\big]+\varepsilon v_rv_j\big[(n-2)\beta\alpha_k-\alpha T_l\big],
\end{aligned}
\tag{2.3}
$$

and

$$
\begin{aligned}
\nabla_rR&=a_rR+\varepsilon\big\{[2(n^2+n-2)\alpha+2(n-1)\beta v_av^a]\alpha_r+2(n+1)\alpha v_av^a\beta_r\\
&\quad-[n(n-1)\alpha^2+2\alpha\beta(n-1)v_av^a]a_r+4(n+1)\alpha\beta v_av^at_r\\
&\quad+4[(n-1)\beta\alpha_av^a-(n-1)\beta^2\sigma v_av^a-\alpha T_av^a]v_r\big\}.
\end{aligned}
\tag{2.4}
$$

Now, let us consider the covariant derivative of the conformal curvature tensor C_{ijkl} of the hypersurface (M,g):

$$
\begin{aligned}
\nabla_rC_{ijkl}&=\nabla_rR_{ijkl}-\frac{1}{n-2}(g_{jk}\nabla_rR_{il}-g_{jl}\nabla_rR_{ik}+g_{il}\nabla_rR_{jk}-g_{ik}\nabla_rR_{jl})\\
&\quad+\frac{1}{(n-1)(n-2)}\nabla_rR(g_{jk}g_{il}-g_{jl}g_{ik}).
\end{aligned}
$$

Substituting (2.2), (2.3) and (2.4) into this relation, we obtain, after some calculation

$$
\begin{aligned}
\nabla_rC_{ijkl}&=a_rC_{ijkl}+\frac{4\alpha}{(n-1)(n-2)}\varepsilon(v_av^aT_r-T_av^av_r)(g_{il}g_{jk}-g_{ik}g_{jl})\\
&\quad-\frac{2\alpha}{n-2}\varepsilon T_r(g_{il}v_jv_k+g_{jk}v_iv_l-g_{ik}v_jv_l-g_{jl}v_iv_k)
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{\alpha}{n-2}\varepsilon\big[(g_{ri}g_{jk}-g_{rj}g_{ik})(T_av^av_l-v_av^aT_l)\\
&\quad+(g_{rj}g_{il}-g_{ri}g_{jl})(T_av^av_k-v_av^aT_k)\\
&\quad+(g_{rk}g_{il}-g_{rl}g_{ik})(T_av^av_j-v_av^aT_j)\\
&\quad+(g_{rl}g_{jk}-g_{rk}g_{lj})(T_av^av_i-v_av^aT_i)\big]\\
&+\frac{\alpha}{n-2}\varepsilon v_r\big[g_{jk}(v_iT_l+v_lT_i)+g_{il}(v_jT_k+v_kT_j)\\
&\quad-g_{jl}(v_iT_k+v_kT_i)-g_{ik}(v_jT_l+v_lT_j)\big]\\
&+\varepsilon\alpha\big[g_{ri}v_j(v_kT_l-v_lT_k)+g_{rj}v_i(v_lT_k-v_kT_l)\\
&\quad+g_{rk}v_l(v_iT_j-v_jT_i)+g_{rl}v_k(v_jT_i-v_iT_j)\big].
\end{aligned}
$$

If $\alpha=0$, (2.5) reduces to

$$\nabla_r C_{ijkl}=a_rC_{ijkl}, \tag{2.6}$$

i.e. the hypersurface is conformally recurrent or (in the case $a_r=0$) conformally symmetric.

If $\alpha\neq 0$, the condition (2.6) is satisfied if and only if

$$
\begin{aligned}
&\frac{4}{n-1}(v_av^aT_r-T_av^av_r)(g_{il}g_{jk}-g_{ik}g_{jl})\\
&\quad-2T_r(g_{il}v_jv_k+g_{jk}v_iv_l-g_{ik}v_jv_l-g_{jl}v_iv_k)\\
&\quad+(g_{ri}g_{jk}-g_{rj}g_{ik})(T_av^av_l-v_av^aT_l)+(g_{rj}g_{il}-g_{ri}g_{jl})(T_av^av_k-v_av^aT_k)\\
&\quad+(g_{rk}g_{il}-g_{rl}g_{ik})(T_av^av_j-v_av^aT_j)+(g_{rl}g_{jk}-g_{rk}g_{lj})(T_av^av_i-v_av^aT_i)\\
&\quad+v_r\big[g_{jk}(v_iT_l+T_iv_l)+g_{il}(v_jT_k+v_kT_j)-g_{jl}(v_iT_k+v_kT_i)-g_{ik}(v_jT_l+v_lT_j)\big]\\
&\quad(n-2)\big[g_{ri}v_j(v_kT_l-v_lT_k)+g_{rj}v_i(v_lT_k-v_kT_l)\\
&\qquad+g_{rk}v_l(v_iT_j-v_jT_i)+g_{rl}v_k(v_jT_i-v_iT_j)\big]=0.
\end{aligned}
$$

Transvecting this relation with v^iv^l, we find

$$
\begin{aligned}
&-\frac{2}{n-1}(g_{jk}v_bv^b-v_jv_k)(v_av^aT_r-T_av^av_r)\\
&+(g_{rj}v_av^a-v_rv_j)(v_bv^bT_k-T_bv^bv_k)+(g_{rk}v_av^a-v_rv_k)(v_bv^bT_j-T_bv^bv_j)=0.
\end{aligned}
$$

Transvecting the last relation with g^{rj}, we get

$$\frac{(n+1)(n-2)}{n-1}v_bv^b(v_av^aT_k-T_av^av_k)=0,$$

or

$$T_k=\frac{T_av^a}{v_bv^b}v_k,$$

because $n>3$ and $v_av^a\neq 0$.

Conversely, if $T_k=fv_k$, where f is a scalar function, (2.5) reduces to (2.6). Thus we have

THEOREM 1. *Let* $(\overline{M}, \overline{g})$ *be a recurrent Riemannian space with* a_ρ *as a recurrence vector field. Let* (M, g) *be its quasiumbilical hypersurface satisfying* (1.6). *Then*:

if $\alpha = 0$, (M, g) *is a conformally recurrent manifold with* $a_r = a_\rho B_r^\rho$ *as a recurrence vector field*;

if $\alpha \neq 0$, (M, g) *is a conformally recurrent space if and only if* $T_k = f v_k$, *where* f *is a scalar function.*

In the following section we shall show that in the case $\alpha \neq 0$, $T_k = f v_k$, the additional informations about the hypersurface (M, g) can be obtained.

3. The case $\alpha \neq 0$, $\alpha n + \beta v_a v^a \neq 0$. In the case $T_k = f v_k$ the Codazzi equation (1.8) reduces to

$$\overline{R}_{\alpha\beta\gamma\delta} N^\alpha B_j^\beta B_k^\gamma B_l^\delta = g_{jk} w_l - g_{jl} w_k. \tag{3.1}$$

Applying the operator ∇_r to (3.1) and using (1.10) and (1.11) we get

$$\begin{aligned} g_{jk} \nabla_r w_l - g_{jl} \nabla_r w_k &= \nabla_\rho \overline{R}_{\alpha\beta\gamma\delta} N^\alpha B_j^\beta B_k^\gamma B_l^\delta B_r^\rho \\ &- \alpha \overline{R}_{\alpha\beta\gamma\delta} B_r^\alpha B_j^\beta B_k^\gamma B_l^\delta - \beta v_r v^t \overline{R}_{\alpha\beta\gamma\delta} B_t^\alpha B_j^\beta B_k^\gamma B_l^\delta \\ &- \varepsilon(\alpha g_{rk} + \beta v_r v_k) \overline{R}_{\alpha\beta\gamma\delta} N^\alpha B_j^\beta B_l^\delta N^\gamma + \varepsilon(\alpha g_{rl} + \beta v_r v_l) \overline{R}_{\alpha\beta\gamma\delta} N^\alpha B_j^\beta B_k^\gamma N^\delta. \end{aligned}$$

Substituting (2.1) and (3.1) into this relation, putting

$$\overline{R}_{\alpha\beta\gamma\delta} N^\alpha B_j^\beta B_l^\delta N^\gamma = P_{jl}, \tag{3.2}$$

and taking into account (1.7), we find

$$\begin{aligned} \alpha R_{rjkl} &= -\beta v_r v^a R_{ajkl} + \varepsilon\alpha^3 (g_{rl} g_{jk} - g_{rk} g_{jl}) \\ &+ \varepsilon\alpha^2 \beta (g_{rl} v_j v_k + g_{jk} v_r v_l - g_{rk} v_j v_l - g_{jl} v_r v_k) \\ &+ g_{jk} (\varepsilon\alpha^2 \beta v_l v_r + \varepsilon\alpha\beta^2 v_a v^a v_r v_l + a_r w_l - \nabla_r w_l) \\ &- g_{jl} (\varepsilon\alpha^2 \beta v_k v_r + \varepsilon\alpha\beta^2 v_a v^a v_r v_k + a_r w_k - \nabla_r w_k) \\ &- \varepsilon(\alpha g_{rk} + \beta v_r v_k) P_{jl} + \varepsilon(\alpha g_{rl} + \beta v_r v_l) P_{jk}. \end{aligned} \tag{3.3}$$

Substituting (1.12), we have

$$\begin{aligned} \alpha R_{rjkl} &= \varepsilon\alpha^3 (g_{rl} g_{jk} - g_{rk} g_{jl}) \\ &+ \varepsilon\alpha^2 \beta (g_{rl} v_j v_k + g_{jk} v_r v_l - g_{rk} v_j v_l - g_{jl} v_r v_k) \\ &+ g_{jk} [\varepsilon(\alpha^2 \beta + \alpha\beta^2 v_a v^a) v_r v_l + \beta v_r S_l + a_r w_l - \nabla_r w_l] \\ &- g_{jl} [\varepsilon(\alpha^2 \beta + \alpha\beta^2 v_a v^a) v_r v_k + \beta v_r S_k + a_r w_k - \nabla_r w_k] \\ &+ \beta v_r v_j M_{lk} - \varepsilon(\alpha g_{rk} + \beta v_r v_k) P_{jl} + \varepsilon(\alpha g_{rl} + \beta v_r v_l) P_{jk}. \end{aligned}$$

Interchanging the indices r and j and also k and l and adding the obtained relation to the preceding one, we get

$$2\alpha R_{rjkl} = 2\varepsilon\alpha^3 (g_{rl} g_{jk} - g_{rk} g_{jl})$$

$$
\begin{aligned}
&+ \varepsilon(3\alpha^2\beta + \alpha\beta^2 v_a v^a)(g_{rl}v_j v_k + g_{jk}v_r v_l - g_{rk}v_j v_l - g_{jl}v_r v_k) \\
&+ g_{jk}(\beta v_r S_l + a_r w_l - \nabla_r w_l) - g_{jl}(\beta v_r S_k + a_r w_k - \nabla_r w_k) \\
&+ g_{rl}(\beta v_j S_k + a_j w_k - \nabla_j w_k) - g_{rk}(\beta v_j S_l + a_j w_l - \nabla_j w_l) \\
&- \varepsilon(\alpha g_{rk} + \beta v_r v_k)P_{jl} + \varepsilon(\alpha g_{rl} + \beta v_r v_l)P_{jk} \\
&- \varepsilon(\alpha g_{jl} + \beta v_j v_l)P_{rk} + \varepsilon(\alpha g_{jk} + \beta v_j v_k)P_{rl}.
\end{aligned}
$$

Now, we replace the indices r, j, k, l with l, k, j, r respectively. Adding the obtained relation to the preceding one and taking into account that P_{ij} is a symmetric tensor, we find

$$
\begin{aligned}
4\alpha R_{rjkl} &= 4\varepsilon\alpha^3(g_{rl}g_{jk} - g_{rk}g_{jl}) \\
&+ 2\varepsilon(3\alpha^2\beta + \alpha\beta^2 v_a v^a)(g_{rl}v_j v_k + g_{jk}v_r v_l - g_{rk}v_j v_l - g_{jl}v_r v_k) \\
&+ g_{jk}[\beta(v_r S_l + v_l S_r) + a_r w_l + a_l w_r - (\nabla_r w_l + \nabla_l w_r)] \\
&- g_{jl}[\beta(v_r S_k + v_k S_r) + a_r w_k + a_k w_r - (\nabla_r w_k + \nabla_k w_r)] \\
&+ g_{rl}[\beta(v_j S_k + v_k S_j) + a_j w_k + a_k w_j - (\nabla_j w_k + \nabla_k w_j)] \\
&- g_{rk}[\beta(v_j S_l + v_l S_j) + a_j w_l + a_l w_j - (\nabla_j w_l + \nabla_l w_j)] \\
&- 2\varepsilon(\alpha g_{rk} + \beta v_r v_k)P_{jl} + 2\varepsilon(\alpha g_{rl} + \beta v_r v_l)P_{jk} \\
&- 2\varepsilon(\alpha g_{jl} + \beta v_j v_l)P_{rk} + 2\varepsilon(\alpha g_{jk} + \beta v_j v_k)P_{rl}.
\end{aligned} \tag{3.4}
$$

Now we have to determine P_{ij}. To do this, we interchange the indices r and j in (3.3) and add the obtained relation to (3.3). We get

$$
\begin{aligned}
&- \beta v_r v^a R_{ajkl} - \beta v_j v^a R_{arkl} \\
&+ g_{jk}[\varepsilon(\alpha^2\beta + \alpha\beta^2 v_a v^a)v_l v_r + a_r w_l - \nabla_r w_l] \\
&- g_{jl}[\varepsilon(\alpha^2\beta + \alpha\beta^2 v_a v^a)v_k v_r + a_r w_k - \nabla_r w_k] \\
&+ g_{rk}[\varepsilon(\alpha^2\beta + \alpha\beta^2 v_a v^a)v_j v_l + a_j w_l - \nabla_j w_l] \\
&- g_{rl}[\varepsilon(\alpha^2\beta + \alpha\beta^2 v_a v^a)v_j v_k + a_j w_k - \nabla_j w_k] \\
&- \varepsilon(\alpha g_{rk} + \beta v_r v_k)P_{jl} + \varepsilon(\alpha g_{rl} + \beta v_r v_l)P_{jk} \\
&- \varepsilon(\alpha g_{jk} + \beta v_j v_k)P_{rl} + \varepsilon(\alpha g_{jl} + \beta v_j v_l)P_{rk} = 0.
\end{aligned}
$$

This, in view of (1.12), can be written in the form

$$
\begin{aligned}
&2\beta v_r v_j M_{lk} \\
&\quad + g_{jk}[\varepsilon(\alpha^2\beta + \alpha\beta^2 v_a v^a)v_l v_r + a_r w_l + \beta v_r S_l - \nabla_r w_l] \\
&\quad - g_{jl}[\varepsilon(\alpha^2\beta + \alpha\beta^2 v_a v^a)v_k v_r + a_r w_k + \beta v_r S_k - \nabla_r w_k] \\
&\quad + g_{rk}[\varepsilon(\alpha^2\beta + \alpha\beta^2 v_a v^a)v_j v_l + a_j w_l + \beta v_j S_l - \nabla_j w_l] \\
&\quad - g_{rl}[\varepsilon(\alpha^2\beta + \alpha\beta^2 v_a v^a)v_j v_k + a_j w_k + \beta v_j S_k - \nabla_j w_k] \\
&\quad - \varepsilon(\alpha g_{rk} + \beta v_r v_k)P_{jl} + \varepsilon(\alpha g_{rl} + \beta v_r v_l)P_{jk} \\
&\quad - \varepsilon(\alpha g_{jk} + \beta v_j v_k)P_{rl} + \varepsilon(\alpha g_{jl} + \beta v_j v_l)P_{rk} = 0.
\end{aligned}
$$

Transvecting this with g^{kr}, we have

$$\begin{aligned}&\varepsilon(n\alpha^2\beta + n\alpha\beta^2 v_a v^a + \beta P)v_j v_l + n a_j w_l + n\beta v_j S_l - n\nabla_j w_l\\ &\quad - g_{jl}[\varepsilon(\alpha^2\beta + \alpha\beta^2 v_a v^a)v_b v^b - \varepsilon\alpha P + w_a a^a + \beta S_a v^a - \nabla_a w^a] + 2\beta v_j v^a M_{la}\\ &= \varepsilon(\alpha n + \beta v_a v^a)P_{jl} + \varepsilon\beta(v_j v^a P_{al} - v_l v^a P_{ja}),\end{aligned}\tag{3.5}$$

where $P = P_{ab}g^{ab}$. This function can be determined as follows.

Taking into account (1.3), we get from (3.2)

$$P = \overline{R}_{\alpha\gamma}N^\alpha N^\gamma.$$

On the other hand, transvecting (1.7) with g^{il} and using (1.3) and (3.2), we find

$$\overline{R}_{\beta\gamma}B^\beta_j B^\gamma_k - \varepsilon P_{jk} = R_{jk} - \varepsilon[(n-1)\alpha^2 + \alpha\beta v_a v^a]g_{jk} - \varepsilon(n-2)\alpha\beta v_j v_k,$$

from which, transvecting with g^{jk}, we get

$$P = \frac{\varepsilon}{2}(\overline{R} - R) + \frac{n(n-1)}{2}\alpha^2 + (n-1)\alpha\beta v_a v^a.\tag{3.6}$$

Now, let us return to (3.5). Interchanging the indices j and l in (3.5) and subtracting the obtained relation from (3.5), we find

$$\begin{aligned}&\varepsilon\beta(v_j v^a P_{al} - v_l v^a P_{ja}) = \beta(v_j v^a M_{la} - v_l v^a M_{ja})\\ &+ \frac{1}{2}n(a_j w_l - a_l w_j) + \frac{1}{2}n\beta(v_j S_l - v_l S_j) - \frac{n}{2}(\nabla_j w_l - \nabla_l w_j).\end{aligned}$$

Substituting this into (3.5), we find

$$\begin{aligned}&\varepsilon(\alpha n + \beta v_a v^a)P_{jl} = \varepsilon(n\alpha^2\beta + \alpha\beta^2 v_a v^a + \beta P)v_j v_l\\ &+ \frac{n}{2}(a_j w_l + a_l w_j) + \frac{n}{2}\beta(v_j S_l + v_l S_j) - \frac{n}{2}(\nabla_j w_l + \nabla_l w_j)\\ &- g_{jl}[\varepsilon(\alpha^2\beta + \alpha\beta^2 v_a v^a)v_b v^b - \varepsilon\alpha P + w_a a^a + \beta S_a v^a - \nabla_a w^a]\\ &+ \beta(v_j v^a M_{la} + v_l v^a M_{ja}).\end{aligned}$$

Thus, supposing

$$\alpha n + \beta v_a v^a \neq 0,$$

we have

$$\begin{aligned}P_{jl} = C\left[v_j\left(\frac{n}{2}S_l + v^a M_{la}\right) + v_l\left(\frac{n}{2}S_j + v^a M_{ja}\right)\right]&\\ + D[a_j w_l + a_l w_j - (\nabla_j w_l + \nabla_l w_j)] + E v_j v_l + F g_{jl},&\end{aligned}\tag{3.7}$$

where we have put

$$C = \frac{\beta}{\alpha n + \beta v_a v^a}, \qquad D = \frac{n}{2(\alpha n + \beta v_a v^a)},\tag{3.8}$$

$$
\begin{aligned}
E &= \frac{n\alpha\beta(\alpha+\beta v_a v^a)+\beta P}{\alpha n+\beta v_a v^a},\\
F &= \frac{\alpha P-\alpha\beta(\alpha+\beta v_a v^a)v_b v^b-\varepsilon w_b a^b-\varepsilon\beta S_a v^a+\varepsilon\nabla_a w^a}{\alpha n+\beta v_a v^a}
\end{aligned}
\tag{3.9}
$$

Substituting (3.7) into (3.4), we get

$$
\begin{aligned}
R_{rjkl} &= A(g_{rl}g_{jk}-g_{rk}g_{jl})\\
&+ g_{rl}(Bv_jv_k+v_jp_k+v_kp_j+LQ_{jk})+g_{jk}(Bv_rv_l+v_rp_l+v_lp_r+LQ_{rl})\\
&- g_{rk}(Bv_jv_l+v_jp_l+v_lp_j+LQ_{jl})-g_{jl}(Bv_rv_k+v_rp_k+v_kp_r+LQ_{rk})\\
&+ N(v_jv_kQ_{rl}+v_rv_lQ_{jk}-v_rv_kQ_{jl}-v_jv_lQ_{rk}),
\end{aligned}
\tag{3.10}
$$

where we have put

$$
A=\varepsilon(\alpha^2+F), \qquad B=\frac{\varepsilon}{2}\left(3\alpha\beta+\beta^2 v_a v^a+E+\frac{\beta}{\alpha}F\right) \tag{3.11}
$$

$$
p_l = S_l\left(\frac{\beta}{\alpha}+\varepsilon nC\right)+2\varepsilon C v^a M_{la} \tag{3.12}
$$

$$
L=\frac{1}{\alpha}+2\varepsilon D, \quad N=2\varepsilon\frac{\beta}{\alpha}D, \quad Q_{rl}=a_rw_l+a_lw_r-\nabla_rw_l-\nabla_lw_r. \tag{3.13}
$$

Thus, we have

THEOREM 2. *Let $(\overline{M},\overline{g})$ be a recurrent Riemannian space. Let (M,g) be its quasiumbilical hypersurface satisfying* (1.6). *If, in addition,*

$$
\alpha\neq 0, \qquad n\alpha+\beta v_a v^a\neq 0, \qquad T_k=fv_k,
$$

then the curvature tensor of the hypersurface can be expressed in the form (3.10), *where the functions A and B are given by* (3.11) *and* (3.9), *the functions L and N — by* (3.13) *and* (3.8), *while p_i is an 1-form determined by* (3.12), (3.8) *and* (1.13). *As for the function P, it is given by* (3.6).

If $p_l=0$ and $Q_{ij}=0$, (3.10) reduces to

$$
R_{rjkl}=A(g_{rl}g_{jk}-g_{rk}g_{jl})+B(g_{rl}v_jv_k+g_{rk}v_rv_l-g_{rk}v_jv_l-g_{jl}v_rv_k),
$$

i.e. the hypersurface is a space of quasi-constant curvature. For example, if $\alpha=\text{const.}\neq 0$, and $\beta=0$, the conditions $p_l=w_l=0$, according to (1.9), (3.12) and (3.8), are satisfied. The condition $w_l=0$ implies $Q_{ij}=0$. On the other hand, if $\beta=0$, then $B=0$ and the hypersurface is a space of constant curvature. Thus

THEOREM 3. *Umbilical hypersurface of the recurrent Riemannian space satisfying $\alpha=\text{const.}\neq 0$ is a space of constant curvature.*

As a consequence of Theorem 3, we have (c.f. [**5**, Theorem 1])

COROLLARY. *Umbilical hypersurface of a Cartan-symmetric Riemannian space satisfying $\alpha=\text{const.}\neq 0$, is a space of constant curvature.*

We get from (3.10)

$$\begin{aligned} R_{jk} &= g_{jk}[(n-1)A + 2p_a v^a + LQ_{ab}g^{ab} + Bv_a v^a] \\ &\quad + v_j v_k[(n-2)B + NQ_{ab}g^{ab}] + Q_{jk}[(n-2)L + v_a v^a N] \\ &\quad + (n-2)(p_j v_k + p_k v_j) - N(v_k v^a Q_{ja} + v_j v^a Q_{ak}), \end{aligned} \tag{3.14}$$

$$\begin{aligned} R &= n(n-1)A + 4(n-1)p_a v^a + 2(n-1)PQ_{ab}g^{ab} + 2(n-1)Bv_a v^a \\ &\quad + 2N(Q_{ab}g^{ab}v_p v^p - Q_{ab}v^a v^b). \end{aligned} \tag{3.15}$$

Substituting (3.10), (3.14) and (3.15) into the expression

$$\begin{aligned} C_{rjkl} &= R_{rjkl} - \frac{1}{n-2}(g_{jk}R_{rl} - g_{jl}R_{rk} + g_{rl}R_{jk} - g_{rk}R_{jl}) \\ &\quad + \frac{R}{(n-1)(n-2)}(g_{jk}g_{rl} - g_{jl}g_{rk}) \end{aligned}$$

of the conformal curvature tensor, we find

$$\begin{aligned} C_{rjkl} = N\Big\{ &\frac{2}{(n-1)(n-2)}(g_{jk}g_{rl} - g_{jl}g_{rk})(Q_{ab}g^{ab}v_p v^p - Q_{ab}v^a v^b) \\ &+ Q_{rl}v_j v_k + Q_{jk}v_r v_l - Q_{jl}v_r v_k - Q_{rk}v_j v_l \\ &+ \frac{g_{jk}}{n-2}(v_r v^a Q_{al} + v_l v^a Q_{ar} - v_a v^a Q_{rl} - Q_{ab}g^{ab}v_r v_l) \\ &- \frac{g_{jl}}{n-2}(v_r v^a Q_{ak} + v_k v^a Q_{ar} - v_a v^a Q_{rk} - Q_{ab}g^{ab}v_r v_k) \\ &+ \frac{g_{rl}}{n-2}(v_k v^a Q_{ja} + v_j v^a Q_{ak} - v_a v^a Q_{jk} - Q_{ab}g^{ab}v_j v_k) \\ &- \frac{g_{rk}}{n-2}(v_j v^a Q_{la} + v_l v^a Q_{ja} - v_a v^a Q_{jl} - Q_{ab}g^{ab}v_j v_l)\Big\}. \end{aligned}$$

Thus, we have

Corrollary. *Let* $(\overline{M}, \overline{g})$ *be a recurrent Riemannian space and* (M, g) *its quasiumbilical hypersurface satisfying* (1.6). *If, in addition,*

$$\alpha \neq 0, \qquad \alpha n + \beta v_a v^a \neq 0, \qquad T_k = f v_k, \qquad Q_{ij} = 0,$$

(M, g) *is conformally flat.*

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