PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série tome 47 (61), 1990, 88–98

LINEAR OPTIMAL CONTROL PROBLEM IN PLANE

Vladimir Janković

Abstract. The linear optimal time control problem in two-dimensional phase space is investigated. The control set is convex compact subset of the phase space, and the class of admissible controls is the class of all controls with at most countably many discontinuities, all of them being of the first kind. Necessary and sufficient conditions, a uniqueness theorem and the existence theorem are proved.

1. Introduction

We shall investigate the following optimal control problem:

The control set U is an arbitrary convex compact set in $\mathbf{R}^2$. The admissible control $u(\cdot)$ is a function which maps an interval of the real line into the control set U, with at most countably many discontinuities, all of them being of the first kind.

The phase space X is a two-dimensional Euclidean space $\mathbf{R}^2$.

Let A be a bounded linear operator mapping the phase space X into itself. Continuous function $x(\cdot):[t_0,t_1]\to X$ is said to be a trajectory which corresponds to the admissible control $u(\cdot):[t_0,t_1]\to U$ if it is differentiable at each point $t\in[t_0,t_1]$ at which the control $u(\cdot)$ is continuous and

$$\dot{x}(t)=Ax(t)+u(t).$$

It is easy to show that at every discontinuity point of the admissible control $u(\cdot)$ the corresponding phase trajectory has left and right derivatives.

Classical theory concerning the existence of solutions of linear differential equations transfers to the previous equation, although its right hand side is not continuous. This is because the right hand side is Riemann integrable and so the equation is equivalent to the integral equation.

$$x(t)=x(\tau)+\int_\tau^t [Ax(s)+u(s)]\,ds.$$

AMS Subject Classification (1985): Primary 49 A 10, 49 B 10

Key words and phrases: Control set, optimal control, maximum principle.

Classical theory is obtained studying this integral equation. So, if $\tau \in [t_0, t_1]$ and $\xi \in X$, then the unique trajectory $x(\cdot) : [t_0, t_1] \to X$ satisfying the condition $x(\tau) = \xi$ exists. It can be represented in the form

$$x(t) = R(t,\tau)\left[\xi + \int_\tau^t R(\tau, s)u(s)\,ds\right],$$

where $R(t,\tau)$ is the resolvent of the homogeneous linear differential equation $\dot{x} = Ax$.

Denote by x_0 and x_1 two points of the phase space X. The admissible control $u(\cdot) : [t_0, t_1] \to U$ accomplishes the passage of the phase point from the position x_0 to the position x_1 if the corresponding trajectory has the beginning at the point x_0 and the end at x_1. The difference $t_1 - t_0$ is called the passage time.

We shall study the problem of minimization of pasage time. The admissible control $\widehat{u}(\cdot)$ and the corresponding trajectory $\widehat{x}(\cdot)$ are optimal if they accomplish the pasage of the phase point from the position x_0 into the position x_1 in the shortest time.

It is easy to see that, without loss of generality, we can suppose that $x_1 = 0$.

In [**1**], [**2**], [**3**], [**4**], [**5**] and [**6**] several versions of the linear time control problem are investigated, with the class of admissible controls being either the class of piecewise continuous controls, or the class of measurable controls. The linear time control problem with variable end points is studied in [**7**] where the class of admissible controls is given axiomatically in the similar way as the class of admissible controls for the general optimal control problem is given in [**2**]. In this article we show that, in the case when the dimension of the phase space is two and the control set is an arbitrary convex compact set, it is most natural to choose for the class of admissible controls the class of all controls with countably many discontinuities of the first kind only.

In Section 2 a maximum principle is proved to be a necessary condition for optimality for the linear control problem in plane. The idea of the proof is the same as in [**1**], and that theorem could be derived from some earlier theorems, for example from [**7**]. The proof is given here for the sake of completeness.

In Section 3 the condition of strict stability is introduced. It is shown that the maximum principle becomes a sufficient condition for optimality if the condition of strict stability is satisfied. This statement is a generalization of the similar statements from [**1**], [**2**] and [**3**]. In [**4**], a similar statement is proved under the condition of strong stability. The condition of strong stability is such that checking its validity by definition requires the knowledge of the solution of linear time control problem that we investigate. In [**4**], a proposition giving a sufficient condition for strong stability is proved. The assumptions of that proposition are stronger then the condition of strict stability.

In Section 4 a generalization of the theorem about the finite number of switchings (in two-dimensional case) is proved and a uniqueness theorem for optimal control is derived from it. A similar theorem corresponding to the case when the phase space has arbitrary dimension is proved in [**7**]. In the two-dimensional case the conclusion we obtain is more precise.

In Section 5 the existence theorem for optimal control is proved. This theorem could be derived from the existence theorems proved in [**5**] and [**7**]. However, a new proof is given here, obtained by the application of an idea similar to that in [**3**]. Contrary to the proofs in [**5**] and [**7**] this proof is relatively elementary and gives the possibility to develop a numerical method, similar to that in [**3**].

In Section 6 it is shown that, under certain restrictions, the whole concept developed in the previous sections can be aplied to the linear time control problem in the space of dimension grater than two.

2. A necessary condition for optimality

LEMMA 2.1. *If $u(\cdot) : [t_0, t_1] \to U$ is an admissible control with the corresponding trajectory $x(\cdot) : [t_0, t_1] \to X$ and $p(\cdot) : [t_0, t_1] \to X^*$ is a solution of the differential equation $\dot{p} = -pA$, then*

$$\int_{t_0}^{t_1} p(t)u(t)\,dt = p(t_1)x(t_1) - p(t_0)x(t_0).$$

Proof. The assertion of the lemma follows from

$$\begin{aligned}\frac{d}{dt}p(t)u(t) &= \dot{p}(t)x(t) + p(t)\dot{x}(t)\\ &= -p(t)Ax(t) + p(t)Ax(t) + p(t)u(t)\\ &= p(t)u(t). \ \square\end{aligned}$$

The admissible control $u(\cdot) : [t_0, t_1] \to U$ is extremal if a non-trivial solution $p(\cdot) : [t_0, t_1] \to X^*$ of the differential equation $\dot{p} = -pA$, such that the condition

$$\max_{u \in U} p(t)u = p(t)u(t)$$

(the maximum condition) is satisfied for each point t in which the control $u(\cdot)$ is continuous.

THEOREM 2.1. (Maximum principle). *If the admissible control $\widehat{u}(\cdot) : [t_0, t_1] \to U$ is optimal, then it is extremal.*

Proof. The sphere of accessibility $S(T)$, $T \geq 0$, is the set of points of the phase space X which could be transferred to the position 0 in time T by some admissible control. Let us prove that every sphere of accessibility is a convex set. Let $T > 0$ and let x_1 and x_2 be two points from $S(T)$. There exist admissible controls $u_1(\cdot)$ and $u_2(\cdot)$ which transfer the phase points from the positions x_1 and x_2, respectively, to 0. Denote by $x_1(\cdot)$ and $x_2(\cdot)$ the corresponding trajectories. Denote by x a point laying on the line segment $[x_1, x_2]$. Take number λ, $0 \leq \lambda \leq 1$, such that

$$x = (1 - \lambda)x_1 + \lambda x_2.$$

The trajectory

$$x(t) = (1-\lambda)x_1(t) + \lambda x_2(t)$$

corresponds to the admissible control

$$u(t) = (1-\lambda)u_1(t) + \lambda u_2(t).$$

Indeed,

$$\begin{aligned}
\dot{x}(t) &= (1-\lambda)\dot{x}_1(t) + \lambda\dot{x}_2(t) \\
&= (1-\lambda)[Ax_1(t) + u_1(t)] + (1-\lambda)[Ax_2(t) + u_2(t)] \\
&= A[(1-\lambda)x_1(t) + \lambda x_2(t)] + [(1-\lambda)u_1(t) + \lambda u_2(t)] \\
&= Ax(t) + u(t).
\end{aligned}$$

Since

$$x(0) = (1-\lambda)x_1(0) + \lambda x_2(0) = (1-\lambda)x_1 + \lambda x_2 = x,$$

and

$$x(T) = (1-\lambda)x_1(T) + \lambda x_2(T) = (1-\lambda)0 + \lambda 0 = 0,$$

the control $u(\,\cdot\,)$ transfers the phase point from the position x to 0. It follows that $x \in S(T)$.

Let us prove that x_0 is a boundary point of the sphere of accessibility $S(T)$, where $T = t_1 - t_0$. Suppose that this is not true, i.e. that x_0 is an interior point of the sphere of accessibility $S(T)$. There exists a triangle with vertices $x_1, x_2, x_3 \in S(T)$, having x_0 as an interior point. Let $u_1(\,\cdot\,)$, $u_2(\,\cdot\,)$, $u_3(\,\cdot\,)$ be admissible controls and let $x_1(\,\cdot\,)$, $x_2(\,\cdot\,)$, $x_3(\,\cdot\,)$ be the corresponding trajectories, defined on $[0,T]$ and transferring the phase points from the positions x_1, x_2 and x_3, respectively, to 0. For sufficiently small $\tau > 0$ the triangle with vertices $x_1(\tau)$, $x_2(\tau)$ and $x_3(\tau)$ differs a little from the triangle with vertices x_1, x_2 and x_3, and therefore it contains the point x_0 too. Since points $x_1(\tau)$, $x_2(\tau)$, and $x_3(\tau)$ belong to the sphere of accessibility $S(T-\tau)$, and since it is convex, then the point x_0 belongs to the sphere of accessibility $S(T-\tau)$. This is contrary to the assumption that the phase point cannot be transferred from the position x_0 to the position 0 in the time which is shorter than T.

Since the point x_0 lays on the boundary of the sphere of accessibility $S(T)$, there exists $\widehat{p}_0 \in X^*$ such that $\widehat{p}_0 x \geq \widehat{p}_0 x_0$ for each $x \in S(T)$. Let $\widehat{p}(\,\cdot\,) : [t_0, t_1] \to X^*$ be the solution of the differential equation $\dot{p} = -pA$ which satisfies the condition $\widehat{p}(t_0) = \widehat{p}_0$. Suppose that the maximum condition is not satisfied. Then there exists a point $\tau \in [t_0, t_1]$ at which the optimal control $\widehat{u}(\,\cdot\,)$ is continuous and the vector $u \in U$ such that

$$\widehat{p}(\tau)u > \widehat{p}(\tau)\widehat{u}(\tau).$$

Because of the continuity an interval I, $\tau \in I \subseteq [t_0, t_1]$, exists such that

$$\widehat{p}(t)u > \widehat{p}(t)\widehat{u}(t)$$

for every $t \in I$. Let $u(\,\cdot\,) : [t_0, t_1] \to U$ be the admissible control defined by

$$u(t) = \begin{cases} \widehat{u}(t) & t \in [t_0, t_1] \setminus I, \\ u & t \in I \end{cases}$$

and let $x(\,\cdot\,)$ be the corresponding trajectory ending at 0. From

$$\int_{t_0}^{t_1} \widehat{p}(t)u(t)\,dt = \widehat{p}(t_1)x(t_1) - \widehat{p}(t_0)x(t_0) = -\widehat{p}_0 x(t_0),$$

$$\int_{t_0}^{t_1} \widehat{p}(t)\widehat{u}(t)\,dt = \widehat{p}(t_1)\widehat{x}(t_1) - \widehat{p}(t_0)\widehat{x}(t_0) = -\widehat{p}_0 x_0,$$

$$\int_{t_0}^{t_1} \widehat{p}(t)u(t)\,dt > \int_{t_0}^{t_1} \widehat{p}(t)\widehat{u}(t)\,dt,$$

it follows that

$$\widehat{p}_0 x(t_0) < \widehat{p}_0 x_0.$$

On the other hand, since $x(t_0) \in S(T)$, then

$$\widehat{p}_0 x(t_0) \geq \widehat{p}_0 x_0.$$

Contradiction! □

3. A sufficient condition for optimality

Lemma 3.1. *Let $p(\,\cdot\,) : [t_0, t_1] \to X^*$ be a nontrivial solution of the differential equation $\dot{p} = -pA$ and let $w \neq 0$ be a vector from the phase space X. If $p(t)w = 0$ for infinitely many values t from the interval $[t_0, t_1]$, then w is an eigenvector of the operator A.*

Proof. The set of points of the interval $[t_0, t_1]$ for which $p(t)w = 0$ has at least one accumulation point. Let τ be one of them. The equality $p(\tau)w = 0$ is valid because of the continuity of the function $p(t)w$. The function $p(t)w$ has the derivative

$$\frac{d}{dt} p(t)w = -p(t)Aw.$$

Since between every two roots of a differentiable function there exists at least one root of its derivative, τ is an accumulation point of the roots of the function $p(t)Aw$. Because of the continuity, the equality $p(\tau)Aw = 0$ is valid. Vectors w and Aw are parallel because they belong to the kernel of the functional $p(\tau)$. □

We say that the condition of strict stability is fulfilled if 0 is the interior point of the set U, or if 0 is the boundary point of U, and besides, the unique support line of the set U at the point 0 exists, and that line is not invariant with respect to the operator A.

Theorem 3.1. *Let the condition of strict stability be fulfilled. If the admissible control $\widehat{u}(\,\cdot\,) : [t_0, t_1] \to U$ transferring the phase point from x_0 to 0, is extremal, then it is optimal.*

Proof. Let $\widehat{x}(\,\cdot\,) : [t_0, t_1] \to X$ be the trajectory corresponding to the admissible control $\widehat{u}(\,\cdot\,)$, starting at x_0 and terminating at 0, and let $\widehat{p}(\,\cdot\,) : [t_0, t_1] \to X^*$ be the nontrivial solution of the differential equation $\dot{p} = -pA$ which the condition

$$\max_{u \in U} \widehat{p}(t)u = \widehat{p}(t)\widehat{u}(t)$$

is fulfilled for each point t in which the control $\widehat{u}(\cdot)$ is continuous. Let us suppose that there exist the admissible control $u(\cdot)$ and the corresponding trajectory $x(\cdot)$ defined on the interval $[t_0, t_2]$, $t_0 < t_2 < t_1$, and transferring the phase point from x_0 to 0. Since

$$\int_{t_0}^{t_1} \widehat{p}(t)\widehat{u}(t)\,dt = \widehat{p}(t_1)\widehat{x}(t_1) - \widehat{p}(t_0)\widehat{x}(t_0) = -\widehat{p}(t_0)x_0,$$
$$\int_{t_0}^{t_2} \widehat{p}(t)u(t)\,dt = \widehat{p}(t_1)x(t_1) - \widehat{p}(t_0)x(t_0) = -\widehat{p}(t_0)x_0,$$

then

$$\int_{t_0}^{t_1} \widehat{p}(t)\widehat{u}(t)\,dt = \int_{t_0}^{t_2} \widehat{p}(t)u(t)\,dt.$$

Since

$$\widehat{p}(t)\widehat{u}(t) = \max_{u\in U} \widehat{p}(t)u \geq \widehat{p}(t)u(t),$$

then

$$\int_{t_0}^{t_2} \widehat{p}(t)\widehat{u}(t)\,dt \geq \int_{t_0}^{t_2} \widehat{p}(t)u(t)\,dt.$$

From the previous relations we obtain that

$$\int_{t_2}^{t_1} \widehat{p}(t)\widehat{u}(t)\,dt \leq 0.$$

Since $0 \in U$, then

$$\widehat{p}(t)\widehat{u}(t) = \max_{u\in U} \widehat{p}(t)u \geq \widehat{p}(t)0 = 0$$

for each point $t \in [t_2, t_1]$ at which the control $\widehat{u}(\cdot)$ is continuous. From the two previous relations it follows that

$$\max_{u\in U} \widehat{p}(t)u = 0$$

for each $t \in [t_2, t_1]$. This is possible only in the case when 0 belongs to the boundary of the set U and when $\widehat{p}(t)w = 0$ for every $t \in [t_2, t_1]$, where w is the vector parallel to the support line of the set U at the point 0. It follows that w is an eigenvector of the operator A, i.e. that the support line of the set U at the point 0 is invariant with respect to the operator A. Contradiction! □

4. Uniqueness of the extremal control

We say that the condition of general position is fulfilled if the control set U has no support line which is parallel to an eigenvector of the operator A and has more than one point in common with U.

THEOREM 4.1. *Let the condition of general position be fulfilled. If $p(\cdot) : [t_0, t_1] \to X^*$ is a nontrivial solution of the differential equation $\dot{p} = -pA$, then the unique admissible control $u(\cdot) : [t_0, t_1] \to U$, satisfying the condition*

$$\max_{u\in U} p(t)u = p(t)u(t),$$

exists. (We shall not distinguish between the admissible controls taking different values at points of discontinuity).

Proof. Let $\{w_i \mid i \in I\}$ be a set of vectors parallel to support lines of the control set U having more than one point in common with U. That set is countable and no vector w_i, $i \in I$, is an eigenvector of the operator A. According to Lemma 3.1, the equality $p(t)w_i = 0$ is satisfied only for finitely many values $t \in [t_0, t_1]$. Therefore, the maximum condition uniquely defines the control $u(\cdot)$ in all but at most countably many points of the interval $[t_0, t_1]$.

Let τ be a point of uniqueness of control $u(\cdot)$. Then the functional $p(\tau)$ reaches its maximum on the set U at the unique point $u(\tau)$. Let $\varepsilon > 0$. The function

$$u \mapsto \frac{p(\tau)[u - u(\tau)]}{\|u - u(\tau)\|}$$

is continuous and negative on the compact set $U \setminus B]u(\tau), \varepsilon[$. Therefore there exists $\mu > 0$ such that

$$p(\tau)[u - u(\tau)] \le -\mu\|u - u(\tau)\|,$$

for every $u \in U \setminus B]u(\tau), \varepsilon[$. There exists $\delta > 0$, such that

$$\|p(t) - p(\tau)\| < \mu,$$

for $|t - \tau| < \delta$. Suppose $|t - \tau| < \delta$. For $u \in U \setminus B]u(\tau), \varepsilon[$ we have that

$$\begin{aligned} p(t)u &= [p(t) - p(\tau)][u - u(\tau)] + p(\tau)[u - u(\tau)] + p(t)u(\tau) \\ &< \mu\|u - u(\tau)\| - \mu\|u - u(\tau)\| + p(t)u(\tau) = p(t)u(\tau). \end{aligned}$$

Since the functional $p(t)$ reaches its maximum on the set U at the point $u(t)$, then $u(t) \in B]u(\tau), \varepsilon[$. Hence, the control $u(\cdot)$ is continuous at τ.

Let τ be a point at which the control $u(\cdot)$ is not uniquely defined by the maximum condition. Then the set of points at which the functional $p(\tau)$ reaches its maximum on the set U is the segment $[u_-, u_+]$ which lies on the border of the set U. We can suppose that vectors $p(\tau)$ and $u_+ - u_-$ form a positive oriented angle. Let us prove that the control $u(\cdot)$ has a limit as $t \to \tau_+$. Denote by $\varphi(t)$ the angle between vectors $p(\tau)$ and $p(t)$. Obviously, the function $\varphi(\cdot)$ is continuous and $\varphi(\tau) = 0$. As $t \to \tau_+$, $\varphi(t)$ tends to 0 from one side. If the contrary were true, the equality $p(t)(u_+ - u_-) = 0$ would be satisfied for infinitely many values t (for every t for which $\varphi(t) = 0$), which is contrary to the assumption that the condition of general position is satisfied. Suppose, for example, that $\varphi(t) \to 0_+$ as $t \to \tau_+$. Let us prove that then $u_+(\tau) = u_+$. Let $\varepsilon > 0$. Let $\overline{u} \in [u_-, u_+]$. The function

$$u \mapsto \frac{p(\tau)[u - \overline{u}]}{\|u - \overline{u}\|}$$

is continuous and negative on the compact set $U \setminus B][u_-, u_+], \varepsilon[$. Therefore, a $\mu > 0$ exists such that

$$p(\tau)[u - \overline{u}] \le -\mu\|u - \overline{u}\|,$$

for every $u \in U \setminus B][u_-, u_+], \varepsilon[$. There exists $\delta > 0$, such that

$$\|p(t) - p(\tau)\| < \mu,$$

for $|t - \tau| < \delta$. Suppose $|t - \tau| < \delta$. For $u \in U \setminus B][u_-, u_+], \varepsilon[$ we have that

$$\begin{aligned} p(t)u &= [p(t) - p(\tau)][u - \overline{u}] + p(\tau)[u - \overline{u}] + p(t)\overline{u} \\ &< \mu\|u - \overline{u}\| - \mu\|u - \overline{u}\| + p(t)\overline{u} = p(t)\overline{u}. \end{aligned}$$

Since the functional $p(t)$ reaches its maximum on the set U at the point $u(t)$, then $u(t) \in B][u_-, u_+], \varepsilon[$ for $\tau < t < \tau + \delta$. For δ sufficiently small we have $0 < \varphi(t) < \pi/2$ when $\tau < t < \tau + \delta$. The line which is orthogonal to $p(t)$ and goes through the point $u(t)$ and the line determined by the segment $[u_-, u_+]$ are the support lines of the set U. The angle determined by those lines in which the set U lies, is obtuse. The point $u(t)$ lies on one side of that angle and the points u_- and u_+ on the other side. Besides, the point u_+ is closer to the vertex of that angle then the point u_-. Therefore, the closest point of the segment $[u_-, u_+]$ to the point $u(t)$ is u_+. It follows that $u(t) \in B]u_+, \varepsilon[$ for $\tau < t < \tau + \delta$. □

THEOREM 4.2. *Let the condition of general position be satisfied. Every two optimal controls defined on the same interval coincide.*

Proof. Let $\widehat{u}(\cdot)$ and $u(\cdot)$ be optimal controls defined on the same interval $[t_0, t_1]$. According to the theorem 3.1, there exists a nontrivial solution $\widehat{p}(\cdot)$ of the differential equation $\dot{p} = -pA$, such that

$$\max_{u \in U} \widehat{p}(t)u = \widehat{p}(t)\widehat{u}(t),$$

when $\widehat{u}(\cdot)$ is continuous at the point t. Denote by $\widehat{x}(\cdot)$ and $x(\cdot)$ the trajectories corresponding to the controls $\widehat{u}(\cdot)$ and $u(\cdot)$, beginning at the point x_0 and ending at 0. From

$$\begin{aligned} \int_{t_0}^{t_1} \widehat{p}(t)\widehat{u}(t)\,dt &= \widehat{p}(t_1)\widehat{x}(t_1) - \widehat{p}(t_0)\widehat{x}(t_0) = -\widehat{p}(t_0)x_0, \\ \int_{t_0}^{t_1} \widehat{p}(t)u(t)\,dt &= \widehat{p}(t_1)x(t_1) - \widehat{p}(t_0)x(t_0) = -\widehat{p}(t_0)x_0 \end{aligned}$$

and

$$\widehat{p}(t)\widehat{u}(t) \geq \widehat{p}(t)u(t),$$

it follows that

$$\widehat{p}(t)\widehat{u}(t) = \widehat{p}(t)u(t).$$

According to the previous theorem we conclude that the controls $\widehat{u}(\cdot)$ and $u(\cdot)$ coincide. □

5. Existence of optimal control

THEOREM 5.1. *Suppose conditions of strict stability and general position are satisfied. If the admissible control transferring the phase point from the position x_0 into* 0 *exists, then the optimal control exists.*

Proof. Let $T \geq 0$ and let $\pi \in X^* \setminus \{0\}$. Denote by $p(t,\pi)$ the solution of the differential equation $\dot{p} = -pA$, which satisfies the initial condition $p(0) = \pi$. From the linear differential equation theory it is known that $p(t,\pi) = \pi R(0,T)$. Denote by $u(t,\pi)$ the control defined on $[0,+\infty[$ with countably many discontinuities of the first kind only, satisfying the condition

$$\max_{u\in U} p(t,\pi)u = p(t,\pi)u(t,\pi)$$

at every point of continuity t. Let $x(t,T,\pi)$ be the trajectory defined on the interval $[0,T]$, corresponding to the admissible control $u(t,\pi)$ and ending at 0. Let $\xi(T,\pi)$ be the beginning of that trajectory. Since

$$x(T,T,\pi) = R(T,0)\left[\xi(T,\pi) + \int_0^T R(0,t)u(t,\pi)\,dt\right]$$

then

$$\xi(T,\pi) = -\int_0^T R(0,t)u(t,\pi)\,dt.$$

Let us prove that the function ξ is continuous. Let $\overline{T} \geq 0$, $\overline{\pi} \in X^* \setminus \{0\}$. Then let $\bar{t} \geq 0$ be a point at which the function $u(t,\overline{\pi})$ is uniquely defined by the maximum condition. Using the technique from the proof of Theorem 4.1, it is easy to prove that $u(\bar{t},\pi) \to u(\bar{t},\overline{\pi})$ as $\pi \to \overline{\pi}$. According to the Lebesgue's bounded convergence theorem, we have that

$$\begin{aligned}&|\xi(T,\pi) - \xi(\overline{T},\overline{\pi})|\\ &= \left|\int_0^{\overline{T}} R(0,t)u(t,\pi)\,dt - \int_0^{\overline{T}} R(0,t)u(t,\overline{\pi})\,dt\right| + \left|\int_T^{\overline{T}} R(0,t)u(t,\pi)\,dt\right| \to 0\end{aligned}$$

as $(T,\pi) \to (\overline{T},\overline{\pi})$. It follows that the function ξ is continuous at the point $(\overline{T},\overline{\pi})$.

According to Theorem 3.1 we conclude that $u(t,\pi)$ is the optimal control transferring the phase point from the position $\xi(T,\pi)$ to 0 during the time T. Therefore the point $\xi(T,\pi)$ belongs to the border of the sphere of accessibility $S(T)$. Let x be a point of the sphere of accessibility $S(T)$, different from the point $\xi(T,\pi)$. Let $u(\,\cdot\,) : [0,T] \to U$ be the admissible control transferring the phase point from x to 0 and let $x(\,\cdot\,)$ be the corresponding trajectory. Then

$$p(t,\pi)u(t,\pi) \geq p(t,\pi)u(t)$$

and the inequality is strict on at least one subinterval of $[0,T]$. It follows that

$$\int_0^T p(t,\pi)u(t,\pi)\,dt > \int_0^T p(t,\pi)u(t)\,dt.$$

Since

$$\int_0^T p(t,\pi)u(t,\pi)\,dt = -\pi\xi(T,\pi)$$

and

$$\int_0^T p(t,\pi)u(t)\,dt = -\pi x,$$

we have that $\pi\xi(T,\pi) < \pi x$. Therefore, $\xi(T,\pi) + \ker\pi$ is the support line of the sphere of accessibility $S(T)$ having with it only the point $\xi(T,\pi)$ in common.

Let us prove that $S(T)$ is a compact set. Let Π be the unit sphere of the space X^*. Since the function ξ is continuous and the set Π is compact, then the set $\xi(T,\Pi)$ is compact. It follows that the set $\operatorname{conv}\xi(T,\Pi)$ is compact too. Suppose that there exists a point x from the set $S(T)$ which does not belong to the set $\operatorname{conv}\xi(T,\Pi)$. There exists a functional $\pi \in \Pi$ which takes greater values on the set $\operatorname{conv}\xi(T,\Pi)$ than at the point x. From the previous considerations we conclude that $\pi\xi(T,\pi) < \pi x$. Contradiction! Thus we have proved that $S(T) \subseteq \operatorname{conv}\xi(T,\Pi)$. The converse inclusion follows from $\xi(T,\Pi) \subseteq S(T)$ and the convexity of $S(T)$.

Let us prove that from each border point of the accessibility sphere $S(T)$ the optimal trajectory runs to 0. It will be sufficient to prove that $\operatorname{bd} S(T) = \xi(T,\Pi)$. We already know that $\xi(T,\Pi) \subseteq \operatorname{bd} S(T)$. Let us prove the converse inclusion. Let $x \in \operatorname{bd} S(T)$. The functional $\pi \in \Pi$ that reaches its minimum on the set $S(T)$ at the point x exists. Since $\pi\xi(T,\pi) \leq \pi x$, where the equality is valid only in the case when $x = \xi(T,\pi)$, we conclude that $x = \xi(T,\pi)$.

The half-line L starting at 0 and going through x_0 intersects the border of the sphere of accessibility $S(T)$ at exactly one point $s(T)$. Let $\sigma(T) = \|s(T)\|$. Let us prove that the function σ is continuous. Let $\overline{T} \geq 0$ and $\varepsilon > 0$. The set $L \setminus B]s(\overline{T}),\varepsilon[$ is closed and disjoint from $\operatorname{bd} S(\overline{T})$. Therefore the distance d between those sets is positive. Because of the continuity of the function ξ and the compactness of the set Π, there exists $\delta > 0$ such that $\|\xi(T,\pi) - \xi(\overline{T},\pi)\| < d$ for every $\pi \in \Pi$ and for every $T \geq 0$ which satisfies the condition $\|T - \overline{T}\| < \delta$. It follows that $s(T) \in B]s(\overline{T}),\varepsilon[$, and consequently $\|\sigma(T) - \sigma(\overline{T})\| < \varepsilon$ for every $T \geq 0$ satisfying $\|T - \overline{T}\| < \delta$. That means that the function σ is continuous.

Since the admissible control transferring the phase point from the position x_0 to 0 exists, there exists $T > 0$ such that $\sigma(T) \geq \|x_0\|$. On the other hand, we have $\sigma(0) = 0 \leq \|x_0\|$. Because of the continuity of σ, there exists $T \geq 0$ such that $\sigma(T) = \|x_0\|$, i.e. such that $x_0 \in \operatorname{bd} S(T)$. □

6. Case when phase space dimension is greater then two

If we consider the same optimal control problem with the assumption that the phase space X is the Euclidean space $\mathbf{R}^n$ and the control set U is a convex compact subset of $\mathbf{R}^n$, we cannot apply directly the results from the previous sections to this new setting. In vector spaces of dimension greater then two there exist convex compact sets with uncountably many hyperplanes of support having with them more than one point in common. Consequently, if we wish to apply a similar concept, we have to introduce the following changes in the formulation of the problem:

— the control set U is the convex compact set from X with countably many hyperplanes of support having more than one point in common with U,

— control $u(\cdot)$ is admissible if it has countably many discontinuities.

For such a problem all theorems proved in sections 2, 3, 4 and 5 are valid and they can be proved in the similar way if the conditions of the strict stability and of general position have the following formulation:

— the condition of strict stability is satisfied if 0 is an interior point of the set U, or if 0 is the boundary point of the set U and the intersection of all hyperplanes of support of the set U in 0 is not contained in some proper subspace of X which is invariant under the operator A;

— the control set U is in the general position if there exists a non-empty countable family $\mathcal{S}$ of subspaces of the phase space X, which satisfies the following two conditions:

1. each hyperplane of support of the set U, having with U more than one common point is parallel to some subspace from $\mathcal{S}$;

2. no subspace from $\mathcal{S}$ is contained in the proper subspace of the phase space X, which is invariant under the operator A.

REFERENCES

[**1**] Р. В. Гамкрелидзе, *Теория оптимальных по быстродействию процессов в линейных системах*, Изв. АН СССР, серij матем. **22** (1958), 449–474.

[**2**] Л. С. Понтрягин, В. Г. Болтянский, Р. В. Гамкрелидзе, Е. Ф. Мищенко, *Математическая теория оптимальных процессов*, Физматгиз, Москва, 1961.

[**3**] В. Г. Болтянский, *Математические методы оптимального управления*, Наука, Москва, 1969.

[**4**] В. Г. Болтянский, *Линейная задача оптимального управления*, Дифференциальные уравнения **5** (1969), 783–799.

[**5**] V. Janković, *Existence theorems for some optimal control problems*, Mat. Vesnik **5** (**18**)(**33**) (1981), 361–366.

[**6**] V. Janković, *The linear control problem*, Mat. Vesnik **6** (**19**)(**34**) (1982), 41–47.

[**7**] V. Janković, *The linear optimal control problem with variable endpoints*, Publ. Inst. Math. (Beograd) (N.S.) **45**(**59**) (1989), 133–142.

Matematički fakultet
Studentski trg 16
11000 Beograd, Yugoslavia

(Received 21 05 1989)