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FIXED POINT THEOREMS IN BANACH SPACES

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Abstract. A class of self-maps on Banach spaces which have at least one fixed-point is presented and investigated.

We shall prove three fixed point theorems in Banach spaces.

THEOREM 1. *Let K be a closed and convex subset of a Banach space with the norm $\|x\| = d(x, 0)$ and $f : K \to K$ a mapping which satisfies the condition*

$$d[x, f(x)] + d[y, f(y)] \leq qd(x, y) \tag{1}$$

for all x, y in K, where $2 \leq q < 4$. Then f has at least one fixed point.

Proof. Let x_0 in K be arbitrary and let a sequence $\{x_n\}$ be defined by

$$x_{n+1} = [x_n + f(x_n)]/2 \qquad (n = 0, 1, 2, \dots). \tag{2}$$

For this sequence we have

$$x_n - f(x_n) = 2[x_n - (x_n + f(x_n))/2] = 2(x_n - x_{n+1}) \tag{3}$$

and hence $d[x_n, f(x_n)] = \|x_n - f(x_n)\| = 2d(x_n, x_{n+1})$ $(n = 0, 1, 2, \dots)$. Therefore, for $x = x_{n-1}$ and $y = x_n$ the condition (1) states

$$2d(x_{n-1}, x_n) + 2d(x_n, x_{n+1}) \leq qd(x_{n-1}, x_n).$$

Hence we have that $d(x_n, x_{n+1}) \leq cd(x_{n-1}, x_n)$, where $0 \leq c = (q-2)/2 < 1$, as $2 \leq q < 4$. Therefore, $\{x_n\}$ is a Cauchy sequence in K and hence converges to some $u \in K$. Since

$$d[u, f(x_n)] \leq d(u, x_n) + d[x_n, f(x_n)] = d(u, x_n) + 2d(x_n, x_{n+1}),$$

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we conclude that

$$\lim_{n\to\infty} f(x_n) = u. \tag{4}$$

Now, let us put in (1) $x = u$ and $y = x_n$, and use (3). Then we have

$$d[u, f(u)] + 2d(x_n, x_{n+1}) \leq qd(u, x_n).$$

If now n tends to infinity one has $d[u, f(u)] \leq 0$, which implies $f(u) = u$, and our theorem is established.

Remark. The identity mapping satisfies (1). Therefore, maps which satisfy (1) may have many fixed points.

COROLLARY. *Let K be as in Theorem 1 and let $f : K \to K$ be a mapping which satisfies the condition*

$$d[x, f(y)] + d[y, f(x)] \leq pd(x, y); \qquad 0 \leq p < 2. \tag{5}$$

Then f has a fixed point.

Proof. Using the triangle inequality we have

$$d[x, f(x)] + d[y, f(y)] \leq d(x, y) + d[y, f(x)] + d(y, x) + d[x, f(y)].$$

By (5) we now get

$$d[x, f(x)] + d[y, f(y)] \leq pd(x, y) + 2d(x, y).$$

Hence we conclude that f satisfies (1) with $q = p + 2 < 4$.

THEOREM 2. *Let K be as in Theorem 1 and let $f : K \to K$ be a mapping such that*

$$d[f(x), f(y)] + d[x, f(x)] + d[y, f(y)] \leq qd(x, y) \tag{6}$$

for all x, y in K where $2 \leq q < 5$. Then f has at least one fixed point.

Proof. Consider a sequence $\{x_n\}$ in K defined by (2). For this sequence the equalities (3) and

$$x_n - f(x_{n-1}) = [x_{n-1} + f(x_{n-1})]/2 - f(x_{n-1}) = [x_{n-1} - f(x_{n-1})]/2 \tag{7}$$

hold. Then the inequality

$$d[x_n, f(x_n)] - d[x_n, f(x_{n-1})] \leq d[f(x_{n-1}), f(x_n)] \tag{8}$$

becomes

$$2d(x_n, x_{n+1}) - d(x_{n-1}, x_n) \leq d[f(x_{n-1}), f(x_n)]. \tag{9}$$

Now, if we put in (6) $x = x_{n-1}$ and $y = x_n$, and use (3), then from (6) and (9) we get

$$2d(x_n, x_{n+1}) - d(x_{n-1}, x_n) + 2d(x_{n-1}, x_n) + 2d(x_n, x_{n+1}) \leq qd(x_{n-1}, x_n) \tag{10}$$

and hence $d(x_n, x_{n+1}) \leq ((q-1)/4)d(x_{n-1}, x_n)$. Since $1 \leq q < 5$, it follows that the sequence $\{x_n\}$ is a Cauchy sequence which converges to some $u \in K$. Since $f(x_n)$ also converges to u, from the inequality (6) with $x = u$ and $y = x_n$ we get $d[f(u), u] + d[u, f(u)] \leq 0$ which implies $f(u) = u$ and our theorem is established.

Now we give a general form of the inequality (6) which includes (1) and (6).

THEOREM 3. *Let f be a self-map on K, where K is as in Theorem 1. If there exist real numbers a, b and q such that*

$$0 \leq q + |a| - 2b < 2(a+b); \tag{11}$$

$$ad[f(x), f(y)] + b\{d[x, f(x)] + d[y, f(y)]\} \leq qd(x, y) \tag{12}$$

for all x, y in K, then f has at least one fixed point.

Proof. Consider a sequence $\{x_n\}$ defined by (2) and put in (12) $x = x_{n-1}$ and $y = x_n$. If $a \geq 0$, then by (8) and (12) we obtain, similarly as (10) in Theorem 2, the inequality

$$2ad(x_n, x_{n+1}) - |a|d(x_{n-1}, x_n) + 2b[d(x_{n-1}, x_n) + d(x_n, x_{n+1})] \leq qd(x_{n-1}, x_n), \tag{13}$$

since then $-a = -|a|$. If $a < 0$, than we use the inequality

$$d[x_n, f(x_n)] + d[x_n, f(x_{n-1})] \geq d[f(x_{n-1}), f(x_n)] \tag{14}$$

instead of (8). Then by (14) and (12) (with $x = x_{n-1}$ and $y = x_n$) we obtain (13), because in this case we can write $-|a|$ instead of a. Therefore, (13) holds for all a, b and q. From (13) we get

$$d(x_n, x_{n+1}) \leq kd(x_{n-1}, x_n); \qquad k = (|a| - 2b + q)/(2(a+b)).$$

Since (11) implies $0 \leq k < 1$, it follows that $\{x_n\}$ is a Cauchy sequence and therefore it converges to some $u \in K$. Using (12) with $x = u$ and $y = x_n$ and letting n to tend to infinity we obtain

$$ad[f(u), u] + bd[u, f(u)] \leq 0.$$

Then, as $a + b > 0$, it follows that $f(u) = u$, completing the proof.

REFERENCES

[**1.**] Lj. B. Ćirić, *Quasi-contractions in Banach spaces*, Publ. Inst. Math. (Beograd) (N.S.) **21** (**35**) (1977), 41–48.

[**2.**] M. Edelstein, *A remark on a theorem of M. A. Krasnoselski*, Amer. Math. Monthly **73** (1966), 509–510.

[**3.**] K. Goebel, W. Kirk and T. Shimi, *A fixed point theorem in uniformly convex spaces*, Boll. U.M.I. (4) **7** (1973), 67–75.

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