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# ON COMMON FIXED POINTS IN LOCALLY CONVEX TOPOLOGICAL VECTOR SPACE

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**Abstract.** We state a new common fixed point theorem for a continuous mapping and a multivalued uper semicontinuous mapping in locally convex topological vector spaces, under some additional conditions.

**1. Introduction.** The aim of this short paper is to use results given by Fan [**3**] and Browder [**1**], to prove the existence of a common fixed point for a single-valued mapping and a multivalued mapping in locally convex topological vector spaces.

For this sort of result the reader may be referred to [**2**], [**4**] and [**5**], for example.

Throughout, by $E$ we will denote a Hausdorff topological vector space. Furthermore, if $K$ is a nonempty subset of $E$, then $2^K$ will be the family of all subsets of $K$ and $C(K)$ the family of all nonempty compact convex subsets of $K$.

Before proceeding to the statement of our result, we recall the concepts of algebraic boundary and multivalued upper semicontinuous mapping.

*Definition* 1 [**1**]. Let $C$ be a closed convex subset of $E$. Then a point $x$ of $C$ is said to lie in the algebraic boundary $\delta(C)$ of $C$ if there exists a finite dimensional subspace $F$ of $E$ such that $x$ lies in the boundary of $C \cap F$.

*Definition* 2 [**4**]. A multivalued maping $T : K \to 2^K$, $K \subset E$, is said to be *upper semicontinuous* if for any closed subset $F$ of $K$, $T^{-1}(F) = \{x \in K : T(x) \cap F \neq \varnothing\}$ is a closed subset of $K$.

**2. Main result.** The main result is inspired by an interesting paper by Itoh and Takahashi [**4**].

Next we combine a theorem of Browder [**1**] with the classic Fan's fixed point theorem [**3**], to obtain a common fixed point theorem for a continuous mapping $f$

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of $X$ into $E$ and an upper semicontinuous mapping $T$ of $X$ into $2^X$, where $X$ is a nonempty compact convex subset of $E$.

THEOREM 1. *Let $E$ be a locally convex space and $X$ a nonempty compact convex subset of $E$. Suppose that $f : X \to E$ is a continuous mapping and $T : X \to 2^X$ is an upper semicontinuous mapping, with $T(x) \in C(X)$ for all $x \in X$. Suppose further that the following condition is satisfied*:

($*$) *For each $x \in X$ and for every $u \in \delta(T(x))$ there exists an element $v$ of $T(x)$ and a real number $\lambda > 0$ (both depending on $x$), such that $f(u)-u = \lambda(v-u)$.*

*If $A = \{x \in X : f(x) = x\}$ is convex, then there exists an element $w$ of $X$ such that $w = f(x) \in T(w)$.*

Before we prove the main result (Theorem 1 above), we recall results given by Fan [**4**] and Browder [**1**].

THEOREM 2 [**3**]. *If $K \in C(E)$, $T : K \to 2^K$ is an upper semicontinuous mapping and for any $x \in K$, $T(x) \in C(K)$, then $\{x \in K : x \in T(x)\}$ is nonempty compact.*

*Remark* 1. Theorem 2 is the classic "Fan's fixed point theorem" [**3**].

THEOREM 3 [**1**]. *Let $K \in C(E)$ and $f$ be a continuous mapping of $K$ into $E$. Suppose that for each $u$ in $\delta(K)$, there exists an element $v$ of $K$ and a real number $\lambda$ (both depending on $u$), such that $f(u) - u = \lambda(v - u)$.*

*Then $f$ has a fixed point in $K$.*

*Proof of Theorem* 1. By theorem 3 we observe that, for each $x \in X$, $f$ has a fixed point in $T(x)$. This shows that $x \in X$ implies $T(x) \cap A$ is nonempty. Now, we define a multivalued mapping $S$ of $A$ into $2^A$, by $S(x) = A \cap T(x)$, $x \in A$. Since $X$ is compact and $f$ is continuous in $X$, it follows that $A$ is compact; moreover by hypothesis $A$ is convex. Hence, $S(x)$ is nonempty compact convex, for each $x \in A$.

Furthermore, since $F \subset A$ implies that $S^{-1}(F) = A \cap T^{-1}(F)$, and $T$ is upper semicontinuous, we conclude that $S$ is upper semicontinuous. Thus, by Theorem 2 there exists a fixed point $w$ of $S$ in $A$. For this $w$ we have $f(w) = w \in T(w)$. The proof is now complete.

*Remark* 2. Theorem 1 remains true when the condition ($*$) is replaced by the following one:

($**$) *If $x \in X$ and $u \in \delta(T(x))$, then $f(u) \in T(x)$.*

Indeed, if $x \in X$ and $u \in \delta(T(x))$, then $v = f(u) \in T(x)$ by the condition ($**$). For these $v$ and $\lambda = 1$ we have $f(u) - u = \lambda(v - u)$.

*Remark* 3. Note that this reduces to Fan's fixed point theorem if $f$ is the identity on $X$. In this case if $x \in X$ and $u \in \delta(T(x))$, then the condition ($*$) of the hypothesis of Theorem 1 is satisfied for $v = u$ and $\lambda > 0$, $\lambda$ arbitrary.

*Remark* 4. Condition $(*)$ is not necessary for the conclusion of Theorem 1, since one adds the following hypothesis:

a) $f$ is a single-valued mapping of $X$ into $X$ and

b) $f(T(x)) \subset T(f(x))$ for all $x \in X$ (see [**4**]).

*Remark* 5. In Theorem 1 if $f$ is affine, then $A$ is automatically convex.

## REFERENCES

[**1**] F. E. Browder, *On a new generalization of the Schauder fixed point theorem*, Math. Ann. **174** (1967), 285–290.

[**2**] J. L. C. Camargo, *Teoremas sobre existência de ponto fixo e aplicações*, doctoral dissertation, Instituto Technológico de Aeronáutica (I.T.A.), Brasil, 1982.

[**3**] K. Fan, *Fixed point an minimax theorems in locally convex topological linear spaces*, Proc. Nat. Acad. Sci. USA **38** (1952), 121–126.

[**4**] S. Itoh and W. Takahashi, *Single valued mappings, multivalued mappings and fixed-point theorems*, J. Math. Anal. Appl. **59** (1977), 514–521.

[**5**] S. Itoh and W. Takahashi, *The common fixed point theory of single-valued mappings and multivalued mappings*, Pacific. J. Math. **79** (1978), 493–508.

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