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# ON ULTRAPOWERS OF LINEAR TOPOLOGICAL SPACES WITHOUT CONVEXITY CONDITIONS

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**Abstract.** We define and study the full and the bounded ultrapower of a linear topological space without convexity conditions. We obtain also some results which are true for ultrapowers of locally convex spaces.

## 1. Introduction

The theory of ultraproducts and, particularly, ultrapowers has proved to be an important tool in functional analysis. Let us mention the papers [**4**] and [**5**] in which some interesting results on nonlinear classification of locally convex spaces were obtained by this method. As is well known, a lot of linear topological spaces which are important in applications (e.g. the spaces $L_p$, $0 \leq p < 1$) are not locally convex. So, it is natural to ask to what extent such results can be transfered to the case of non-locally convex linear topological spaces. Although it was remarked in [**4**] that such a transformation is natural, we want to point in this paper to same new moments which arise in this setting.

Ultraproducts of linear topological spaces can be defined either over countably-complete, or over countably-incomplete ultrafilters. The first approach is treated e.g. in [**3**] and [**11**], while the second is covered by a wider bibliography (see e.g. [**4**–**8**]). Each of these approaches has certain advantages — while the first one enables us to consider ultraproducts of different spaces, the second is naturally concerned only with the ultrapower of a certain space. But this latter approach has a greater importance in applications and it does not need any assumptions on the cardinality of the index set. In the present paper we shall follow this approach.

We shall not use any nonstandard theory in this paper, although it is obvious that a lot of ideas and some notation will be inspired by the nonstandard analysis.

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## 2. Preliminaries

The notions concerning the theory of linear topological spaces (*lts*) without convexity conditions can be found in [**1**]. We give here only the basic ones. Let $E$ be a Hausdorff *lts* over the field $K$ (of real or complex numbers). A *string* in $E$ is a sequence $(V_n)_{n\in\mathbf{N}}$ of subsets of $E$ which are circled, absorbing and satisfy $V_{n+1}+V_{n+1}\subset V_n$ $(n=1,2,\dots)$. The string $(V_n)_{n\in\mathbf{N}}$ is said to be *topological* if each $V_n$ is a neighborhood of 0 in $E$. It is clear that each circled neighborhood of 0 in $E$ generates a (non uniquely determined) topological string.

A function $p:E\to\mathbf{R}$ satisfying the conditions:

(a) $p(x)\ge 0$ for each $x\in E$;

(b) $p(x+y)\le p(x)+p(y)$ for each $x,y\in E$;

(c) $p(\lambda x)\le p(x)$ for each $x\in E$ and each $\lambda\in K$, $|\lambda|\le 1$;

(d) if $\lambda_n\in K$, $\lambda_n\to 0$ and $x\in E$, then $p(\lambda_n x)\to 0$

is called an (F)-*seminorm*. If, moreover, $p(x)=0$ implies $x=0$, $p$ is called an (F)-*norm*. (F)-seminorms in a certain sense have the similar role in the theory of linear topological spaces as the seminorms have in the theory of locally convex spaces. Namely, each linear topology on a vector space can be determined by a family of continuous (F)-seminorms. If the *lts* $E$ is metrizable, then its topology can be given by a single (F)-norm. It is known that to each topological string in a *lts* a continuous (F)-seminorm can be corresponded, and conversely.

For the given *lts* $E$ we shall denote: by $\mathcal{U}(E)$ the set of all circled and closed neighborhoods of 0, by $\mathcal{F}(E)$ the set of all continuous (F)-seminorms, by $\mathcal{B}(E)$ the set of all circled and bounded subsets, by $\mathcal{P}(E)$ the set of all circled and precompact subsets, by $\aleph_{\mathcal{U}}(E)$ the least cardinality of a base of neighborhoods of 0, by $\aleph_{\mathcal{B}}(E)$ the least cardinality of a fundamental system of bounded subsets, and $\aleph=\aleph(E)=\max\{\aleph_{\mathcal{U}}(E),\aleph_{\mathcal{B}}(E)\}$.

The rest of the terminology is taken from [**1**] or [**16**]. Particularly, when we say, e.g., "barrelled space", that means "barrelled in the category of *lts*" ("ultra-barrelled" in the terminology of [**9**]).

A filter on the set $I$ is said to be an *ultrafilter* if there is no strictly stronger filter on $I$. The ultrafilter $\mathcal{D}$ on the set $I$ is called *countably-incomplete* if there exists a sequence of sets $I_n\in\mathcal{D}$, with $I_n\supset I_{n+1}$ $(n=1,2,\dots)$ and $\bigcap_{n\in\mathbf{N}}I_n=\varnothing$. For example, each free ultrafilter on the set $\mathbf{N}$ of positive integers is countably-incomplete. For our purposes we need also the notion of an $\aleph$-good ultrafilter ($\aleph$ — an infinite cardinal); its definition and main properties can be found in [**4**], [**13**] or [**17**], so we omit the details here. We mention only that on each set of cardinality not less than $\aleph$ there exists an $\aleph^+$-good countably-incomplete ultrafilter. Also, each countably-incomplete ultrafilter is $\aleph_1$-good.

Let $A$ be an arbitrary set, $I$ an index set and $\mathcal{D}$ an ultrafilter on $I$. Then the *set-theoretical ultrapower* $A^I/\mathcal{D}$ is defined as the Cartesian power $A^I$, factored by the equivalence relation $(a_i)\sim(b_i)\iff\{i: a_i=b_i\}\in\mathcal{D}$. The equivalence class containing $(a_i)$ is denoted by $(a_i)/\mathcal{D}$. If $A_i\subset A$ $(i\in I)$ are given, we write

$\prod_i A_i/\mathcal{D}$ for the set $\{(a_i)/\mathcal{D} \in A^I/\mathcal{D} : a_i \in A_i \,(i \in I)\}$. Such subsets of $A^I/\mathcal{D}$ are called *internal*. If $A_i = B$ for each $i \in I$, we write $B^I/\mathcal{D}$ instead of $\prod_i A_i/\mathcal{D}$.

Some of our results shall depend on certain "goodness" of the given ultrafilter $\mathcal{D}$, i.e. on some saturation properties of the corresponding ultrapower. When considering the ultrapower $E^I/\mathcal{D}$ of an *lts* $E$, we shall always assume that $\mathcal{D}$ is $\aleph^+$-good, where $\aleph = \aleph(E)$, although some of these results can be obtained without this assumption.

## 3. The full and the bounded ultrapower of a linear topological space

Let now $E$ be a Hausdorff *lts* and $E^I/\mathcal{D}$ its set-theoretical ultrapower over a countably-incomplete $\aleph^+$-good ultrafilter $\mathcal{D}$. Identifying $x \in E$ with $(x_i)/\mathcal{D} \in E^I/\mathcal{D}$, $x_i = x$ for each $i \in I$, we shall consider $E$ as a linear subspace of $E^I/\mathcal{D}$.

Following [**4**], [**7**] and [**13**] we have three possibilities for defining finite elements of $E^I/\mathcal{D}$.

1° Finite elements with respect to the topology of the space $E$ form the set

$$\mathrm{fin}(E^I/\mathcal{D}) = \{(x_i)/\mathcal{D} \in E^I/\mathcal{D} : (\forall U \in \mathcal{U}(E))(\exists n \in \mathbf{N})(x_i)/\mathcal{D} \in nU^I/\mathcal{D}\}.$$

2° Finite elements with respect to the corresponding uniform structure of the space $E$ are the elements of the set

$$\begin{aligned}\mathrm{fin}_{\mathcal{V}}(E^I/\mathcal{D}) = \{(x_i)/\mathcal{D} \in E^I/\mathcal{D} : (\forall U \in \mathcal{U}(E))(\exists n \in \mathbf{N})\\ (x_i)/\mathcal{D} \in U^I/\mathcal{D} + \cdots + U^I/\mathcal{D}\,(n \text{ summands})\}.\end{aligned}$$

3° Finite elements with respect to the semi-metrics which define this uniform structure, i.e. with respect to the continuous (F)-seminorms of the space $E$:

$$\mathrm{fin}_{\mathcal{F}}(E^I/\mathcal{D}) = \{(x_i)/\mathcal{D} \in E^I/\mathcal{D} : (\forall p \in \mathcal{F}(E)) \lim_{\mathcal{D}} p(x_i) < +\infty\}.$$

It is clear that the monad can be defined in the two following ways:

1° $\mu(E^I/\mathcal{D}) = \bigcap_{U \in \mathcal{U}(E)} U^I/\mathcal{D}$;

2° $\mu_{\mathcal{F}}(E^I/\mathcal{D}) = \bigcap_{p \in \mathcal{F}(E)} \{(x_i)/\mathcal{D} \in E^I/\mathcal{D} : \lim_{\mathcal{D}} p(x_i) = 0\}$.

In the locally convex case the introduced concepts of finiteness and monads, respectively, are the same. In the general linear topological case the first concept of finiteness is different from the latter two (see Example 1). Before defining the full and the bounded ultrapower of an arbitrary *lts*, let us prove the following

Proposition 3.1. *Let $E$ be a Hausdorff lts and $\mathcal{D}$ a countably-incomplete $\aleph^+$-good ultrafilter. Then*:

(i) *Internal subsets $U^I/\mathcal{D}$ of $E^I/\mathcal{D}$, $U \in \mathcal{U}(E)$, form a base of neighborhoods of* 0 *of a complete topological vector group* (*in the sense of* [**15**]) *which is not Hausdorff. $E$ is a topological subspace of $E^I/\mathcal{D}$.*

(ii) $\mathrm{fin}(E^I/\mathcal{D}) \subset \mathrm{fin}_{\mathcal{V}}(E^I/\mathcal{D}) = \mathrm{fin}_{\mathcal{F}}(E^I/\mathcal{D})$. $\mathrm{fin}(E^I/\mathcal{D})$ *is closed in $E^I/\mathcal{D}$ and it is the maximal subspace on which the topology of $E^I/\mathcal{D}$ induces a linear topology.*

(iii) $\mu(E^I/\mathcal{D}) = \mu_{\mathcal{F}}(E^I/\mathcal{D})$ *and it is a closed subspace of* $E^I/\mathcal{D}$.

*Proof*. (i) For each $U \in \mathcal{U}(E)$ find $V \in \mathcal{U}(E)$ such that $V + V \subset U$ and then $V^I/\mathcal{D} + V^I/\mathcal{D} \subset U^I/\mathcal{D}$. By [**15**] it means that the sets $U^I/\mathcal{D}$, $U \in \mathcal{U}(E)$, form the base of a topological vector group. The completeness follows from the saturation like in [**4**, Theorem 1.1]. Because of $U^I/\mathcal{D} \cap E = U$, $E^I/\mathcal{D}$ induces the initial topology on $E$.

(ii) $\operatorname{fin}(E^I/\mathcal{D}) \subset \operatorname{fin}_{\mathcal{V}}(E^I/\mathcal{D})$ follows from $nU^I/\mathcal{D} \subset U^I/\mathcal{D} + \cdots + U^I/\mathcal{D}$ ($n$ summands). $\operatorname{fin}_{\mathcal{V}}(E^I/\mathcal{D}) = \operatorname{fin}_{\mathcal{F}}(E^I/\mathcal{D})$ follows from the fact (which can be easily checked) that the set $A \subset E$ is bounded with respect to the uniform structure of the space $E$ iff $p(A)$ is bounded for each $p \in \mathcal{F}(E)$. The closedness of $\operatorname{fin}(E^I/\mathcal{D})$ was proved in [**4**] and the maximality follows from the definition.

(iii) Let $(x_i)/\mathcal{D} \in \mu(E^I/\mathcal{D})$ and $p \in \mathcal{F}(E)$. The (F)-seminorm $p$ generates a topological string $(V_k)_{k\in\mathbf{N}}$, where $V_k = \{x \in E : p(x) < 2^{-k}\}$. Now we have $x_i \in V_k$ for each $i \in I$, $k \in \mathbf{N}$, and so $\lim_{\mathcal{D}} p(x_i) \le 2^{-k}$ for each $k \in \mathbf{N}$, which proves that $\lim_{\mathcal{D}} p(x_i) = 0$, i.e., $(x_i)/\mathcal{D} \in \mu_{\mathcal{F}}(E^I/\mathcal{D})$. Conversely, let $(x_i)/\mathcal{D} \notin \mu(E^I/\mathcal{D})$ and so let $U \in \mathcal{U}(E)$ be such that $x_i \notin U$ for each $i \in I$. Form a topological string $(U_n)_{n\in\mathbf{N}}$ in $E$ such that $U = U_1$ and a continuous (F)-seminorm $q$ as in [**1**], p. 11. Then $q(x_i) \ge 1$ for each $i \in I$, which means that $q(x_i) \not\to 0$ ($\mathcal{D}$) i.e. $(x_i)/\mathcal{D} \notin \mu_{\mathcal{F}}(E^I/\mathcal{D})$. □

Now we give an example showing that $\operatorname{fin}(E^I/\mathcal{D})$ can be a proper subspace of $\operatorname{fin}_{\mathcal{V}}(E^I/\mathcal{D})$ and $\operatorname{fin}_{\mathcal{F}}(E^I/\mathcal{D})$.

*Example* 1. Let $E = \{x = (x_n)_{n\in\mathbf{N}} : x_n \in \mathbf{R}, \sum_{n=1}^{\infty} |x_n|^{1/n} < \infty\}$, with the (F)-norm $|x| = \sum_{n=1}^{\infty} |x_n|^{1/n}$. It was proved in [**18**] that there exists a subset $A$ which is bounded with respect to the uniform structure of the space $E$, but not bounded in $E$. It can be easily checked that then there exists $(x_i)/\mathcal{D} \in A^I/\mathcal{D} \subset \operatorname{fin}_{\mathcal{V}}(E^I/\mathcal{D})$ such that $(x_i)/\mathcal{D} \notin \operatorname{fin}(E^I/\mathcal{D})$. (Compare with Example 3.4 in [**7**]).

The results of Proposition 3.1 and Example 1 suggest that the full ultrapower of an arbitrary *lts* $E$ can be defined as the quotient $\operatorname{fin}(E^I/\mathcal{D})/\mu(E^I/\mathcal{D})$ with the corresponding quotient topology. It will be denoted as in [**4**]. So, if $E$ is a Hausdorff *lts* and $\mathcal{D}$ a countably-incomplete $\aleph^+$-good ultrafilter, then

$$(E)_{\mathcal{D}} = \operatorname{fin}(E^I/\mathcal{D})/\mu(E^I/\mathcal{D})$$

is the *full ultrapower of* $E$ *over* $\mathcal{D}$. It is the associated Hausdorff space to the *lts* $\operatorname{fin}(E^I/\mathcal{D})$. Its elements are equivalence classes $(x_i)_{\mathcal{D}} = \pi\big((x_i)/\mathcal{D}\big) = (x_i)/\mathcal{D} + \mu(E^I/\mathcal{D})$, where $\pi$ is the quotient map. The zero-neighborhood basis is formed by the sets $(V)_{\mathcal{D}} = \pi\{V^I/\mathcal{D} \cap \operatorname{fin}(E^I/\mathcal{D})\}$, $V \in \mathcal{U}(E)$.

The bounded ultrapower can be defined as in the locally convex case: first let $\operatorname{bd}(E^I/\mathcal{D}) = \bigcup\{B^I/\mathcal{D} : B \in \mathcal{B}(E)\}$ and then the linear and topological subspace of $(E)_{\mathcal{D}}$ generated by $\pi\big(\operatorname{bd}(E^I/\mathcal{D})\big)$ is denoted by $[E]_{\mathcal{D}}$ and called the *bounded ultrapower of* $E$. Let us mention that, unlike the locally convex case, $\operatorname{bd}(E^I/\mathcal{D})$ is not equal to the set $\{(x_i)/\mathcal{D} \in E^I/\mathcal{D} : (\exists I_0 \in \mathcal{D}) \sup_{I_0} p(x_i) < \infty \text{ for each } p \in \mathcal{F}(E)\}$. It will be proved later (see Proposition 3.3) that $\operatorname{bd}(E^I/\mathcal{D}) \subset \operatorname{fin}(E^I/\mathcal{D})$.

So, we have obtained two new *lts's* which contain the given space $E$ as their subspace.

The following proposition shows that the topology of the full ultrapower can be defined by the family $\mathcal{F}(E)$ of continuous (F)-seminorms of the space $E$, which is similar to the locally convex case.

Proposition 3.2. *If $p \in \mathcal{F}(E)$, then by $\bar{p}\big((x_i)_\mathcal{D}\big) = \lim_\mathcal{D} p(x_i)$ a continuous (F)-seminorm on the space $(E)_\mathcal{D}$ is defined.*

*Proof*. For the proof we need the following result which was in nonstandard terminology obtained in [**7**]: $(x_i)/\mathcal{D} \in \mathrm{fin}(E^I/\mathcal{D})$ iff $(\lambda_i)/\mathcal{D} \cdot (x_i)/\mathcal{D} \in \mu(E^I/\mathcal{D})$ for each $(\lambda_i)/\mathcal{D} \in \mu(\mathbf{R}^I/\mathcal{D})$.

Only the condition (d) from the definition of (F)-seminorms is nontrivial. Let $(x_i)_\mathcal{D} \in (E)_\mathcal{D}$ and let $\lambda_n \in K$, $\lambda_n \to 0$ $(n \to \infty)$. We can take $\lambda_n \in \mathbf{R}$. $\bar{p}$ can be considered as a function from $E^I/\mathcal{D}$ into $\mathbf{R}^I/\mathcal{D}$ and from the continuity of $p$ it follows that $\bar{p}\big(\mu(E^I/\mathcal{D})\big) \subset \mu(\mathbf{R}^I/\mathcal{D})$. So, we have to prove that $\bar{p}\big(\lambda_n(x_i)/\mathcal{D}\big) \to 0$ $(n \to \infty)$, which is equivalent to $\bar{p}\big((\lambda(n_i))/\mathcal{D} \cdot (x_i)/\mathcal{D}\big) \in \mu(\mathbf{R}^I/\mathcal{D})$ for each $(n_i)/\mathcal{D} \in \mathbf{N}^I/\mathcal{D} \setminus \mathbf{N}$. As far as $\big(\lambda(n_i)\big)/\mathcal{D} \in \mu(\mathbf{R}^I/\mathcal{D})$ (because $\lambda_n \to 0$), from the mentioned result it follows that $(\lambda(n_i))/\mathcal{D} \cdot (x_i)/\mathcal{D} \in \mu(E^I/\mathcal{D})$, which gives

$$\bar{p}\big((\lambda(n_i))/\mathcal{D} \cdot (x_i)/\mathcal{D}\big) \in \bar{p}\big(\mu(E^I/\mathcal{D})\big) \subset \mu(\mathbf{R}^I/\mathcal{D})$$

and so the proof is complete. □

The relationship between the spaces $E$, $[E]_\mathcal{D}$, $(E)_\mathcal{D}$ and $\mathrm{fin}(E^I/\mathcal{D})$ can be better understood using the following two subspaces of $E^I/\mathcal{D}$. Since $E^I/\mathcal{D}$ is a complete topological vector group [**15**], the closure $\overline{E}$ of $E$ in $E^I/\mathcal{D}$ has the form:

$$\overline{E} = \big\{(x_i)/\mathcal{D} \in E^I/\mathcal{D} : (\forall U \in \mathcal{U}(E))(\exists y \in E)(x_i)/\mathcal{D} \in (y_i)/\mathcal{D} + U^I/\mathcal{D}, \\ \text{where } y_i = y \text{ for each } i \in I\big\}.$$

We shall denote this space by $\mathrm{pns}(E^I/\mathcal{D})$ [**13**]. On the other hand, we shall denote by $\mathrm{ns}(E^I/\mathcal{D})$ the set of those classes from $E^I/\mathcal{D}$ which have the same image by the cannonical mapping $\pi$ as some points from $E$:

$$\mathrm{ns}(E^I/\mathcal{D}) = \big\{(x_i)/\mathcal{D} \in E^I/\mathcal{D} : (\exists y \in E)(x_i)/\mathcal{D} - (y_i)/\mathcal{D} \in \mu(E^I/\mathcal{D}), \\ \text{where } y_i = y \text{ for each } i \in I\big\}.$$

Now we have the following inclusions between the introduced spaces:

$$\begin{aligned} &\mathrm{ns}(E^I/\mathcal{D}) \subset \mathrm{pns}(E^I/\mathcal{D}) \subset \mathrm{fin}(E^I/\mathcal{D}), \\ &\mu(E^I/\mathcal{D}) \subset \mathrm{pns}(E^I/\mathcal{D}), \\ &\mathrm{bd}(E^I/\mathcal{D}) \subset \mathrm{fin}(E^I/\mathcal{D}) \quad \text{(see the next proposition)}. \end{aligned}$$

Proposition 3.3. *Let $E$ be an arbitrary lts and $\mathcal{D}$ a countably-incomplete $\aleph^+$-good ultrafilter. Then*

(i) *$A \subset E$ is bounded iff $A^I/\mathcal{D} \subset \mathrm{fin}(E^I/\mathcal{D})$*;

(ii) $A \subset E$ *is precompact iff* $A^I/\mathcal{D} \subset \operatorname{pns}(E^I/\mathcal{D})$;

(iii) $A \subset E$ *is relatively compact iff* $A^I/\mathcal{D} \subset \operatorname{ns}(E^I/\mathcal{D})$;

(iv) $E$ *is complete iff* $\operatorname{ns}(E^I/\mathcal{D}) = \operatorname{pns}(E^I/\mathcal{D})$.

*Proof.* (i) and (ii) were proved, using nonstandard analysis in [**7**] and (iii) and (iv) in [**13**]. Let us prove only the "if" part of (i) in a standard way to emphasize the importance of saturation property. If $A \subset E$ is not bounded, then for some $U \in \mathcal{U}(E)$ we can find $n \in \mathbf{N}$ so that $A \not\subset nU$. Now, obviously, the family $\{A^I/\mathcal{D} \cap C(nU)^I/\mathcal{D} : n = 1, 2, \dots\}$ has the finite intersection property, and so the goodness of $\mathcal{D}$ implies that there exists $(x_i)/\mathcal{D} \in A^I/\mathcal{D}$ such that $(x_i)/\mathcal{D} \notin nU^I/\mathcal{D}$, which means that $A^I/\mathcal{D} \not\subset \operatorname{fin}(E^I/\mathcal{D})$. $\square$

*Remark.* The mentioned inclusions between the introduced subsets of $E^I/\mathcal{D}$ and the previous proposition give us the following diagram (the arrows stand for inclusions):

$$\begin{tikzcd} & {[E]_{\mathcal{D}}} \arrow[dr] & \\ E \arrow[ur] \arrow[dr] & & (E)_{\mathcal{D}} \\ & \pi(\operatorname{pns}(E^I/\mathcal{D})) \arrow[ur] & \end{tikzcd}$$

Here $\pi\big(\operatorname{pns}(E^I/\mathcal{D})\big)$ is the completion $\widetilde{E}$ of $E$ and $\widetilde{E}$ and $[E]_{\mathcal{D}}$ are incomparable in general.

One of the natural questions about the relationship between the spaces in this diagram is answered in the following proposition whose proof is nearly the same as in [**4**].

Proposition 3.4. *If $E$ is an lts and $\mathcal{D}$ an $\aleph^+$-good countably incomplete ultrafilter, then the following conditions are equivalent*:

(i) $[E]_{\mathcal{D}}$ *is dense in* $(E)_{\mathcal{D}}$;

(ii) $(\forall \lambda : \mathcal{U}(E) \to \mathbf{R}^+)(\forall V \in \mathcal{U}(E))(\exists \textit{finite}\ \mathcal{U} \subset \mathcal{U}(E))(\exists B \in \mathcal{B}(E))$
$\bigcap_{U \in \mathcal{U}} \lambda(U)U \subset B + V$ (*"the density condition"*).

Let us mention that (non-locally convex) DF-spaces [**1**], according to [**12**], satisfy this condition. Later on (see section 5) we shall prove that Frechet-Montel spaces (also considered in the category of *lts's*) satisfy the density condition, too.

Recall that an *lts* $E$ is called *locally bounded* if it has a bounded neighborhood of 0 and *boundedly compact* if all of its bounded subsets are relatively compact. Locally bounded and metrizable *lts*'s are stable under the full ultrapowers.

The following proposition, which can be compared with [**4**, Proposition 1.5] (see also [**7**] and [**13**]), shows that in a certain sence locally bounded spaces play the same role in the theory of *lts's* as normed spaces do in the theory of locally convex spaces.

Proposition 3.5. *Let $E$ be an lts and $\mathcal{D}$ a countably-incomplete $\aleph^+$-good ultrafilter. Then*:

(i) $\mathrm{fin}(E^I/\mathcal{D}) = \mathrm{bd}(E^I/\mathcal{D})$ *iff* $E$ *is locally bounded. Either of these conditions implies that* $[E]_\mathcal{D} = (E)_\mathcal{D}$ .

(ii) $\mathcal{P}(E) = \mathcal{B}(E)$ *iff* $E$ *is dense in* $[E]_\mathcal{D}$ *and iff* $\mathrm{bd}(E^I/\mathcal{D}) \subset \mathrm{pns}(E^I/\mathcal{D})$.

(iii) $E = [E]_\mathcal{D}$ *iff* $E$ *is boundedly compact.*

(iv) $E$ *satisfies the density condition and* $\mathcal{P}(E) = \mathcal{B}(E)$ *iff* $\pi\big(\mathrm{pns}(E^I/\mathcal{D})\big) = (E)_\mathcal{D}$.

Note that the implication $[E]_\mathcal{D} = (E)_\mathcal{D} \implies \mathrm{fin}(E^I/\mathcal{D}) = \mathrm{bd}(E^I/\mathcal{D})$ does not hold in general (see also Proposition 4.1).

As a corollary we can mention that $E = (E)_\mathcal{D}$ iff $E$ is complete, boundedly compact and satisfies the density condition.

## 4. Ultrapowers and topological operations

In this section we shall deal with some aspects of topological operations with both ultrapowers in the category of *lts's* which differ from the locally convex case.

It can be easily proved that the ultrapower "behaves well" in connection with the projective topology (as defined in the non-locally convex case in [**1**]), i.e., similar propositions as in [**4**, §2] hold. Particularly, the full ultrapower of each *lts* has the projective topology of a system of full ultrapowers of complete metrizable *lts's*. In the proof we have to substitute the fundamental system of absolutely convex neighborhoods of 0 by some fundamental system of topological strings in $E$ (see [**1**]).

When non-locally convex inductive limits are concerned (see definition in [**1**]), there are some differences.

The following proposition, which can be compared with [**4**, Proposition 2.6], shows how some new classes of spaces satisfying $[E]_\mathcal{D} = (E)_\mathcal{D}$ can be obtained. Also, it shows again the role of locally bounded spaces in the category of *lts's*.

Proposition 4.1. *Let* $(E_n)$ *be a sequence of lts's such that each* $E_n$ *is a closed subspace of* $E_{n+1}$. *Let* $E = \operatorname{ind}\lim E_n$ *and let* $\mathcal{D}$ *be a countably-incomplete* $\aleph(E)^+$*-good ultrafilter. Then the following identities hold algebraically*:

$$(E)_\mathcal{D} = \operatorname{ind}\lim (E_n)_\mathcal{D}, \qquad [E]_\mathcal{D} = \operatorname{ind}\lim [E_n]_\mathcal{D}.$$

*If the spaces* $E_n$ *are locally bounded, these identities hold also topologically.*

*Proof*. As in the corresponding proof in [**4**], for the first assertion it is enough to prove that the cannonical injection $\tau : \operatorname{ind}\lim (E_n)_\mathcal{D} \to (E)_\mathcal{D}$ is surjective and this will be proved by establishing the following: for each $y \in \mathrm{fin}(E^I/\mathcal{D})$ there exists $n \in \mathbf{N}$ such that for every $U \in \mathcal{U}(E)$, $y \in E_n^I/\mathcal{D} + U^I/\mathcal{D}$. Let us assume that this is not the case, i.e., for some $y \in \mathrm{fin}(E^I/\mathcal{D})$ and each $n \in \mathbf{N}$ there exists $U_n \in \mathcal{U}(E)$ such that $y \notin E_n^I/\mathcal{D} + U_n^I/\mathcal{D}$. Now we have to use the characteristic shape of zero-neighborhoods in a non-locally convex inductive limit [**1**]. Namely, we can take $U_n = \sum_{k=1}^{\infty} U_{n,k}$* where $U_{n,k} \in \mathcal{U}(E_k)$ for each $n, k \in \mathbf{N}$. Define $V_k \in \mathcal{U}(E_k)$ ($k \in \mathbf{N}$) by $V_k = k^{-1} \bigcap_{j=1}^{k} U_{j,k}$, and we obviously have $V = \sum_{k=1}^{\infty} V_k \in \mathcal{U}(E)$ and

$$V \subset \sum_{k=1}^{n} V_k + \sum_{k=n+1}^{\infty} V_k \subset E_n + \frac{1}{n} \sum_{k=n+1}^{\infty} U_{n,k} \subset \frac{1}{n}(E_n + U_n).$$

Passing to the ultrapowers, we have $y \notin nV^I/\mathcal{D}$, which is a contradiction.

Let now each $E_n$ be a locally bounded space. We have to prove that $\tau^{-1}$ is continuous, and it will be established by proving that $\left(\sum_{n=1}^{\infty} U_n\right)_{\mathcal{D}} \subset \sum_{n=1}^{\infty} (U_n)_{\mathcal{D}}$, where $U_n \in \mathcal{U}(E_n)$ are bounded.

Let $x = (x_i)_{\mathcal{D}} \in \left(\sum_{n=1}^{\infty}\right)_{\mathcal{D}}$, i.e. $(x_i)/\mathcal{D} \in \left(\sum_{n=1}^{\infty} U_n\right)^I/\mathcal{D} \cap \operatorname{fin}(E^I/\mathcal{D})$. By the definition of $\sum_{n=1}^{\infty} U_n$ we have that for some $m \in \mathbf{N}$:

$$x_i = y_{1,i} + y_{2,i} + \cdots + y_{m,i} \qquad \text{where } y_{k,i} \in U_k,\ k = 1, \ldots, m,\ i \in I.$$

The boundedness of the neighborhoods $U_k$ implies that $U_k^I/\mathcal{D} \subset \operatorname{fin}(E_k^I/\mathcal{D}) \subset \operatorname{fin}(E^I/\mathcal{D})$ and so

$$(y_{k,i})/\mathcal{D} \in U_k^I/\mathcal{D} \cap \operatorname{fin}(E_k^I/\mathcal{D}) \subset U_k^I/\mathcal{D} \cap \operatorname{fin}(E^I/\mathcal{D}),$$

hence

$$(x_i)/\mathcal{D} = (y_{1,i})/\mathcal{D} + \cdots + (y_{m,i})/\mathcal{D} \in U_1^I/\mathcal{D} + \cdots + U_m^I/\mathcal{D}.$$

So, $(x_i)_{\mathcal{D}} \in (U_1)_{\mathcal{D}} + \cdots + (U_m)_{\mathcal{D}}$, or $(x_i)_{\mathcal{D}} \in \sum_{n=1}^{\infty} (U_n)_{\mathcal{D}}$, which completes the proof. □

The last proposition gives us also that inductive limits of locally bounded spaces satisfy the density condition.

When the Proposition 2.5 [**4**] is concerned, it can be transfered to the non-locally convex case almost directly. Only the more complicated structure of typical neighborhoods of zero in the direct sum (see [**1**, p. 20]) has to be taken into account.

## 5. (HM)-spaces

From [**17**] we know that a locally convex space $E$ is an (HM)-space if it has the invariant nonstandard hull, i.e., if its full ultrapower $(E)_{\mathcal{D}}$ does not depend on the ultrafilter $\mathcal{D}$. The same definition gives us the notion of arbitrary *lts's* of the type (HM) — we can also say that an *lts* is an (HM)-space if its full ultrapower $(E)_{\mathcal{D}}$ is always isomorphic to the completion $\pi(\overline{E})$ of the space $E$, or, equivalently, if $\operatorname{fin}(E^I/\mathcal{D}) = \operatorname{pns}(E^I/\mathcal{D})$ for each ultrafilter $\mathcal{D}$. So, $E$ is an (HM)-space if $\widetilde{E} = (E)_{\mathcal{D}}$ or, particularly, if $E = (E)_{\mathcal{D}}$. The class of (HM)-spaces is stable under taking subspaces, arbitrary products and projective topologies, but it is not stable under taking separated quotients [**14**]. If the ultrafilter $\mathcal{D}$ is countably complete, then this class is not stable under taking ultrapowers [**3**]. We begin this section with the following

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* $\sum_{k=1}^{\infty} M_k = \bigcup_{N=1}^{\infty} \sum_{k=1}^{N} M_k$

PROPOSITION 5.1. *Let $E$ be an (HM)-space and $\mathcal{D}$ a countably-incomplete $\aleph^+$-good ultrafilter. Then $(E)_\mathcal{D}$ and $[E]_\mathcal{D}$ are also (HM)-spaces.*

*Proof*. Since $E$ is an (HM)-space, its completion is $(E)_\mathcal{D}$ and so the spaces $E$ and $(E)_\mathcal{D}$ have the same full ultrapowers (a space and its dense subspace have the same full ultrapowers). Hence, $(E)_\mathcal{D}$ is an (HM)-space, too. But then its subspace $[E]_\mathcal{D}$ is also an (HM)-space (see [**14**, Theorem 3]). □

*Remark*. So, for (HM)-space we have $\overline{[E]}_\mathcal{D} = (E)_\mathcal{D}$. But the equality $[E]_\mathcal{D} = (E)_\mathcal{D}$ is not obligatory for (HM)-spaces. E.g., if $E$ is an arbitrary Banach space of infinite dimension and $E'_\sigma$ its weak dual, then $[E'_\sigma]_\mathcal{D} = E'_\sigma \neq (E'_\sigma)_\mathcal{D} = [(E'_\sigma)_\mathcal{D}]_\mathcal{D}$. The same example shows that a space and its dense subspace need not have the same bounded ultrapowers.

For *lts's* $E$ and $F$ let $L_\sigma(E,F)$ denotes the space of continuous linear mappings from $E$ to $F$ with the topology of simple convergence. The following proposition can be compared with [**4**, Proposition 1.6].

PROPOSITION 5.2. *Let $E$ and $F$ be lts's such that $F$ is an (HM)-space. Then*

(i) *$L_\sigma(E,F)$ is an (HM)-space.*

(ii) *If, moreover, $E$ is barrelled and $F$ complete, then $[L_\sigma(E,F)]_\mathcal{D} = L_\sigma(E,F)$ for each countably-incomplete, $\aleph^+$-good ultrafilter $\mathcal{D}$, where $\aleph = \aleph(L_\sigma(E,F))$.*

*Proof*. (i) The space $L_\sigma(E,F)$ is a subspace of the space $F^E$ with its product topology [**16**], so the result follows from [**14**].

(ii) By Proposition 3.5, in each (HM)-space the classes of bounded and precompact subsets coincide. From (i) it follows that each simply bounded subset of $L(E,F)$ is precompact. On the other hand, the barrelledness of $E$ implies that each $L_\sigma(E,F)$-bounded subset is equicontinuous [**1**]. So, the closure of such a subset is complete [**16**, III.4.4]. This means that $L_\sigma(E,F)$ is boundedly-compact, which, again by Proposition 3.5, implies that $[L_\sigma(E,F)]_\mathcal{D} = L_\sigma(E,F)$. □

A Banach space is (HM) iff it is of finite dimension, [**7**], [**17**]. For locally bounded *lts's* we have the similar

PROPOSITION 5.3. *A locally bounded space $E$ is (HM) iff it is of finite dimension.*

*Proof*. If $E$ is a locally bounded (HM)-space, then it is clear that $[E]_\mathcal{D} = (E)_\mathcal{D}$ is also a locally bounded (HM)-space. Hence, $(E)_\mathcal{D}$ has a relatively compact neighborhood of 0 and so it is of finite dimension. Then the same is true for $E$. The converse is trivial. □

COROLLARY. *The spaces $l_p$ and $L_p$, $0 \leq p < 1$, are not (HM)-spaces.*

For some classes of locally convex spaces $E$ the condition of being an (HM)-space is equivalent to the condition $\mathcal{B}(E) = \mathcal{P}(E)$ (see [**2**], [**8**]). E.g. a Frechet locally convex space is (HM) iff it is Frechet-Montel. Taking into account that each metrizable *lts* is defined by a single string, i.e. by a single (F)-norm, we have:

Proposition 5.4. *If $E$ is a metrizable lts, then the following conditions are equivalent*:

(i) $\mathcal{P}(E) = \mathcal{B}(E)$;

(ii) *$E$ is an (HM)-space*;

(ii) *the completion $\widetilde{E}$ is a Montel space.*

*Proof*. The implications (ii) $\Longrightarrow$ (ii) $\Longrightarrow$ (i) are trivial. To prove that (i) $\Longrightarrow$ (ii) in our (non-locally convex) case, we cannot use the method of [**8**], Theorem 1, since the boundness is not in general equivalent with the boundness with respect to (F)-seminorms. But, on the other hand, we can use the characterization of (HM)-spaces given in [**7**, Theorem 4.1], which says that the space $E$ is (HM) iff each ultrafilter $\mathcal{F}$ in $E$ satisfying the condition $(\forall U \in \mathcal{U}(E))(\exists n \in \mathbf{N}) nU \in \mathcal{F}$ is a Cauchy ultrafilter. $\square$

The results of the following proposition are partial analogue of the results from [**2**] about locally convex (DF)-spaces. We use the fact from [**12**] that a (non-locally convex) (DF)-space is quasinormed, i.e., it satisfies the density condition. For *lts's* $E$ and $F$ let $L_b(E,F)$ denotes the space of continuous linear mappings from $E$ to $F$ with the topology of uniform convergence on all bounded subsets of $E$. Recall [**9**] that an *lts* $E$ is called *almost-convex* if each bounded subset of $E$ is contained in some bounded, circled and closed subset $B$ such that $B + B \subset \lambda B$ for some $\lambda \geq 0$. E.g., each locally bounded *lts* is almost-convex.

Proposition 5.5. *Let $E$ be a (DF) lts and consider the following conditions*:

(i) *$E$ is an (HM)-space*;

(ii) $\mathcal{P}(E) = \mathcal{B}(E)$;

(iii) *$L_b(E,F)$ is an (HM)-space for each Frechet-Montel lts $F$*;

(iv) *$L_b(E,F)$ is an (HM)-space for each Frechet lts $F$*;

(v) *the completion $\widetilde{E}$ is barrelled.*

*Then* (i) $\Longleftrightarrow$ (ii) *and* (ii) $\Longrightarrow$ (iii). *If $E$ is almost-convex, then* (ii) $\Longrightarrow$ (v) *and* (iv) $\Longrightarrow$ (v).

*Proof*. We know already that (i) $\Longrightarrow$ (ii) holds.

(ii) $\Longrightarrow$ (i): Let $\mathcal{D}$ be an arbitrary countably-incomplete $\aleph^+$-good ultrafilter. Since the (DF)-space $E$ satisfies the density condition we have $\overline{[E]}_{\mathcal{D}} = (E)_{\mathcal{D}}$. Then $\mathcal{P}(E) = \mathcal{B}(E)$, by Proposition 3.5, implies that $\overline{E} = (E)_{\mathcal{D}}$, which means that $E$ is an (HM)-space.

(ii) $\Longrightarrow$ (iii): Under the given assumptions $L_b(E,F)$ is a metrizable space and so it is enough to prove that it has precompact bounded subsets. Suppose that this is not the case, and so that there exists a bounded sequence $(x_n)$ in $L_b(E,F)$, such that $A = \{x_n : n \in \mathbf{N}\}$ is not precompact. But $\mathcal{P}(E) = \mathcal{B}(E)$ implies that the topologies of $L_b(E,F)$ and $L_\sigma(E,F)$ coincide on $A$ (see [**16**, III.4.5]), which by Proposition 5.2, means that $A$ is precompact, hence a contradiction.

(ii) $\Longrightarrow$ (v), (iv) $\Longrightarrow$ (v): Each of the conditions (ii) or (iv) implies that $E$ is quasibarrelled. The first implication was proved in [**1**]. To prove the second consider an arbitrary bounded subset $A$ of $L_b(E,F)$. By (iv) it is precompact and then by [**10**, Proposition 1.3], equicontinuous (in [**10**] $D_b$-spaces were considered which contain the class of (DF)-spaces). Now from [**9**] it follows that the completion $\widetilde{E}$ is barrelled. $\square$

*Remark*. So, as in the locally convex case [**2**], the conditions (i) and (ii) of the last proposition are equivalent for (F) and (DF) spaces in the general situation. For the spaces which are neither (F) nor (DF) we can only say that the space $E$ is (HM) iff $\mathcal{P}(E)=\mathcal{B}(E)$ and $E$ satisfies the density condition (see Proposition 3.5).

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