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ALGEBRA OF ANTISYMMETRIC CHARACTERISTICS

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Abstract. As the basis for derivation of the (P,l)-symmetry groups and reduction of the theory of multiple antisymmetry to the theory of simple antisymmetry, a generalization of antisymmetric characteristic (AC) is used. A catalogue of non-isomorphic AC formed by $1 \leq l \leq 4$ generators is given. The algebraic properties of AC are discussed and the direct product of AC is defined.

During thirty-five years, the theory of multiple antisymmetry, introduced by Zamorzaev in 1953 [**1**] and developed by the Kishinev geometricians (A. F. Palistrant, E. I. Galyarski, P. A. Zabolotni, Yu. S. Karpova and others has acquired the status of a complete theory, extended to all categories of isometric symmetry groups of the space E^n $(n \leq 3)$, different kinds of non-isometric symmetry groups (of similarity symmetry, conformal symmetry, homology ...) and P-symmetry groups ((p)-, (p')-, $(p2)$-symmetry groups). At the same time, it is a very efficient tool for derivation and analysis of multidimensional subperiodic symmetry groups [**2**, **3**].

The idea of antisymmetric characteristic (AC) of the discrete symmetry group G [**4**] makes possible the derivation and partial cataloguing of simple and multiple antisymmetry groups of the M^m-type of any category (e.g. G_2^l, G_{32}^l) [**5**, **6**]. Among the results obtained by use of AC the most important is the complete derivation for $l = 3,4,5$ and partial cataloguing (for $l = \overline{1,6}$) of the Zamorzaev groups G_3^l [**7**, **8**].

In this paper we discuss some different, in certain sense heterogenous questions, integrated by the notion of AC. The main purpose of the paper is not only to give final solutions of particular problems, but also to point out the unsolved problems and designate the potential area of studies related to the idea of AC.

1. Introduction. Let the discrete symmetry group G with a set of generators $\{S_1, \dots, S_r\}$ be given by presentation [**9**]:

$$g_n(S_1, \dots, S_r) = E, \qquad n = 1, \dots, s,$$

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and let $e_1, \dots, e_l$ be antiidentities of the first, ..., l-th kind, which satisfy the relations

$$e_i e_j = e_j e_i, \qquad e_i^2 = E, \qquad e_i S_q = S_q e_i, \qquad i,j = \overline{1,l}, \quad q = \overline{1,r}. \tag{1}$$

The group of (multiple) antisymmetry G' is the combination of G with l antiidentities $e_1, \dots, e_l$ and with their products, which satisfy relations (1). In particular, for $l = i = j = 1$ we have a simple antisymmetry group G'. The effect of antiidentities $e_1, \dots, e_l$ can be interpreted by the system of (geometric or non-geometric) mutually commutative and independent alternating bivalent changes (e.g. $(+,-)$, (black, white) ...), commutative with the generators of the group G. The grups of simple and multiple antisymmetry can be derived by applying the general method of Shubnikov-Zamorzaev [**1**], i.e. by replacing the generators of the group G with antigenerators of one or several independent kinds of antisymmetry. In accordance with the theorem on dividing all groups of simple and multiple antisymmetry into groups of C^k $(1 \le k \le l)$, $C^k M^m$ $(1 \le k, m;\ k+m \le l)$ and M^m $(1 \le m \le l)$ types [**1**] and with the possibility of deriving the groups of the C^k and $C^k M^m$ types directly from the generating group G and from the groups of M^m-type respectively, the only non-trivial problem appears to be that of deriving the groups of M^m-type. This is the reason why in this paper only groups of the M^m-type are discussed.

THEOREM 1 (The existential criterion for groups of M^m-type). *The group of simple or multiple antisymmetry G' is of the M^m-type*

(a) *if all relations given within the presentation of the group G remain satisfied after the generators have been replaced by antigenerators*; *and*

(b) *if the antisymmetry of an arbitrary kind can be derived in the antisymmetry group G' as an independent antisymmetry transformation.*

Definition 1. Let all products of generators of the group G, within which all generators occur at most once, be formed and then separate the subsets of transformations that are equivalent with respect to symmetry. The resulting system is called the antisymmetric characteristic of group G $(AC(G))$ [**4**].

A majority of AC permits reduction, i.e. a transformation into the simplest form. The method for obtaining AC and reduced AC can be illustrated by the example of the symmetry group **pm** of ornaments G_2, given by the set of generators [**1**] $\{a,b\}(m)$. The products of generators formed in accordance with Definition 1 consist of five subsets of transformations equivalent with respect to symmetry, so that we have $AC(\mathbf{pm})$

$$\{m, ma\}\{b\}\{mb, mab\}\{a\}\{ab\}$$

and reduced $AC(\mathbf{pm})$: $\{m, ma\}\{b\}$.

THEOREM 2. *Two groups of simple or multiple antisymmetry G_1' and G_2' of the M^m-type for m fixed, with common generating group G, are equal iff they possess equal AC.*

Every $AC(G)$ completely defines the series $N_m(G)$, where by $N_m(G)$ is denoted the number of groups of the M^m-type derived from G, with m fixed ($1 \leq m \leq l$). For example, $N_1(\mathbf{pm}) = 5$, $N_2(\mathbf{pm}) = 24$, $N_3(\mathbf{pm}) = 84$.

Theorem 3. *Symmetry groups that possess isomorphic AC generate the same number of simple and multiple antisymmetry groups of the M^m-type for every fixed m ($1 \leq m \leq l$), which correspond to each other with regard to structure.*

Corollary. *The derivation of all simple and multiple antisymmetry groups of the M^m-type can be completely reduced to the construction of all non-isomorphic AC and the derivation of simple and multiple antisymmetry groups of the M^m-type from the AC mentioned.*

According to Theorem 3, it is possible to identify every AC with the corresponding isomorphic algebraic term, a representative of the equivalence class which consists of all isomorphic AC. For example, it is possible to identify $AC(\mathbf{pm})$: $\{m, ma\}\{b\}$ with the term $\{A, B\}\{C\}$.

2. The derivation of (P, l)-symmetry groups from P-symmetry groups by use of AC. Let G^p be a junior group of P-symmetry derived from G [**3**]. By replacing in Definition 1 the term "transformations that are equivalent with respect to symmetry" with a more general notion "transformations that are equivalent with respect to P-symmetry", the transition from G to G^p induces the transition from $AC(G)$ to $AC(G^p)$, which makes possible the derivation of groups of (P, l)-symmetry of the M^m-type by use of the metod of AC.

The above can be illustrated by the example of derivation of groups $G_2^{4,l}$, from groups G_2^4: $\{a, b^{(4)}\}(m)$ and $\{a^{(2)}, b^{(4)}\}(m)$. In the first case, in the transition from $G = \mathbf{pm}$ to $G^4 = \{a, b^{(4)}\}(m)$ AC remains unchanged. In the second case, in the transition from $G = \mathbf{pm}$ to $G^p = \{a^{(2)}, b^{(4)}\}(m)$ the equivalence of symmetry transformations is disturbed and the term $\{m, ma^{(2)}\}\{b^{(4)}\}$ is transformed into a new AC: $\{m\}\{ma\}\{b\}$. In accordance with the facts already mentioned, we have

$$\begin{array}{llllll} \{a, b^{(4)}\}(m) & AC: \{m, ma\}\{b\} \cong \{A, B\}\{C\} & N_1 = 5 & N_2 = 24 & N_3 = 84 \\ \{a^{(2)}, b^{(4)}\}(m) & AC: \{m\}\{ma\}\{b\} \cong \{A\}\{B\}\{C\} & N_1 = 7 & N_2 = 42 & N_3 = 168. \end{array}$$

The given numbers N_m denote the number of groups of the M^m-type of the uncomplete $(4, l)$-symmetry. In a general case, besides the numbers N_m for p even, we can discuss also the numbers (N_{m-1}) $(1 \leq m \leq l)$, where by (N_{m-1}) is denoted the number of groups of the complete (p, l)-symmetry of the M^m-type. For p odd, the relation $N_m = (N_m)$ holds, and for p even

$$(N_m) = N_m - (2^m - 1)(N_{m-1}), \qquad (N_0) = 1, \quad 1 \leq m \leq l.$$

One of the most important results of the use of the mentioned method, jointly realised with A. F. Palistrant, is the derivation of the groups $G_3^{p,l}$, from the groups

G_3^p $(p = 3, 4, 6)$ [**10**] and calculation of the numbers N_m and (N_{m-1}) of groups of the category $G_3^{p,l}$;

$$\begin{array}{lllll} N_1 = 4840 & N_2 = 40996 & N_3 = 453881 & N_4 = 5706960 & N_5 = 59996160 \\ (N_1) = 4134 & (N_2) = 29731 & (N_3) = 260114 & (N_4) = 2048760 & (N_5) = 1249920. \end{array}$$

The derivation of (P,l)-symmetry groups of the M^m-type from P-symmetry groups by use of the method of AC can be reduced to a series of successive transitions

$$G \longrightarrow G^P \longrightarrow G^{P,1} \longrightarrow \ldots \longrightarrow G^{P,l}$$

and induced transitions

$$AC(G) \longrightarrow AC(G^P) \longrightarrow AC(G^{P,1}) \longrightarrow \ldots \longrightarrow AC(G^{P,l}).$$

Every induced AC consists of the same number of generators. Since every transition $G^{P,k-1} \longrightarrow G^{P,k}$, $(1 \leq k \leq l)$, is a derivation of simple antisymmetry groups by use of $AC(G^{P,k-1})$, for derivation of all multiple antisymmetry groups, the catalogue of all non-isomorphic AC formed by l generators and simple antisymmetry groups derived by these AC, is completely sufficient.

3. The reduction of the theory of multiple antisymmetry to the theory of simple antisymmetry. The basis of this reduction is the idea already mentioned about the transition $G \longrightarrow G^P$ and induced transition $AC(G) \longrightarrow AC(G^P)$, where $AC(G)$ and $AC(G^P)$ consist of the same number of generators. This means that every step in the derivation of multiple antisymmetry groups

$$G \longrightarrow G^1 \longrightarrow G^2 \longrightarrow \ldots \longrightarrow G^{k-1} \longrightarrow G^k \longrightarrow \ldots \longrightarrow G^l,$$

i.e. the transition $G^{k-1} \longrightarrow G^k$, $(1 \leq k \leq l)$, is a derivation of simple antisymmetry groups by use of $AC(G^{k-1})$, followed by the induced transition $AC(G^{k-1}) \longrightarrow AC(G^k)$, $(1 \leq k \leq l-1)$. All the AC of induced series consist of the same number of generators.

The above can be illustrated by the example of derivation of multiple antisymmetry groups from the symmetry group **pm** of ornaments G_2.

$$\mathbf{pm} \quad \{a,b\}\{m\} \qquad AC: \{m, ma\}\{b\} \cong \{A,B\}\{C\}$$

For $m = 1$, five groups of simple antisymmetry of the M^1-type are obtained:

$$\begin{array}{lll} \{A,B\}\{C\} & \{E,E\}\{e_1\} & \longrightarrow \{A,B\}\{C\} \\ & \{e_1,e_1\}\{E\} & \longrightarrow \{A,B\}\{C\} \\ & \{e_1,e_1\}\{e_1\} & \longrightarrow \{A,B\}\{C\} \\ & \{E,e_1\}\{E\} & \longrightarrow \{A\}\{B\}\{C\} \\ & \{E,e_1\}\{e_1\} & \longrightarrow \{A\}\{B\}\{C\}. \end{array}$$

In the first three cases AC remains unchanged, but in two other cases AC is transformed into the new $AC:\{A\}\{B\}\{C\}$. To continue the derivation of multiple antisymmetry groups of the M^m-type from the symmetry group **pm**, only the derivation of simple antisymmetry groups from AC: $\{A\}\{B\}\{C\}$ is indispensable. This AC is trivial and gives seven groups of simple antisymmetry. If AC: $\{A,B\}\{C\}$ is denoted by 3.2 and AC: $\{A\}\{B\}\{C\}$ by 3.1, then the result obtained can be denoted in a symbolic form by $3.2 \longrightarrow 2(3.1)+3(3.2)$. Then we have

$$\begin{aligned}
N_1(\mathbf{pm}) &= N_1(3.2) = 5 \qquad N_1(3.1) = 7\\
N_2(\mathbf{pm}) &= N_2(3.2) = 2N_1(3.1) + 3N_1(3.2) - 5\cdot 1\\
&= 2(N_1(3.1)-1) + 3(N_1(3.2)-1) = 2\mathbf{6} + 3\mathbf{4}\\
&= 2N_1(3.1) + 3N_1(3.2) - N_1(3.2) = 2N_1(3.1) + 2N_1(3.2) = 24.
\end{aligned}$$

The meaning of every step in the mentioned computation is:

1) subtraction of the number $N_1(3.2)$, i.e. of the five groups of incomplete multiple antisymmetry of the ${}^2M$-type [**7**, **8**];

2) every group of the M^1-type gives exactly one of these ${}^2M$-type groups, so we obtain $2\mathbf{6}+3\mathbf{4}$ groups of complete multiple antisymmetry of the M^2-type [**7**, **8**]. This step contains also essential data for the calculation of the number N_3: **6** groups mentioned possess AC 3.1, two of **4** groups mentioned possess AC 3.1 and two AC 3.1. Among five groups of uncomplete multiple antisymmetry of the ${}^2M$-type there are three groups with AC 3.2 and two with AC 3.1;

3) by substitution $5 = N_1(3.2)$ we obtain $N_2(3.2)$ expressed by $N_1(3.1)$ and $N_1(3.2)$ i.e. $2N_1(3.1) + 2N_1(3.2)$. The sum of coefficients corresponding to the numbers N_1 in the last line gives $N_2(\mathbf{pm}) = 24$.

$$\begin{aligned}
N_3(\mathbf{pm}) &= N_3(3.2) = 2\cdot 6N_1(3.1) + 3\cdot(2N_1(3.1)+2N_1(3.2)) - 24\cdot 3\\
&= 18N_1(3.1) + 6N_1(3.2) - 24\cdot 3 = 18(N_1(3.1)-3) + 6(N_1(3.2)-3)\\
&= 18\mathbf{4} + 6\mathbf{2} = 18N_1(3.1) + 6N_1(3.2) - 3(2N_1(3.1)+2N_1(3.2))\\
&= 12N_1(3.1) = 84 \qquad (N_2(3.2)) = 12.
\end{aligned}$$

Consequently, the method proposed makes possible a complete reduction of the theory of multiple antisymmetry to the theory of simple antisymmetry. This refers not only to the possibility of computation of the numbers N_m and (N_{m-1}), but also to the possibility of applying the method of partial cataloguing of multiple antisymmetry groups of the M^m-type. If we take the advantage of the suggested reduction, the use of this method is considerably simplified and demands only the catalogues of the simple antisymmetry groups of the M^1-type obtained from non-isomorphic AC.

4. The non-isomorphic AC formed by $1 \le l \le 4$ generators. As it is shown in Section 3, the theory of multiple antisymmetry can be reduced to the theory of simple antisymmetry. For that it is necessary to know all non-isomorphic AC formed by l generators. As the result of the study of non-isomorphic AC, the catalogue of those AC formed by $1 \leq l \leq 4$ generators, is obtained. The completeness of this catalogue is proved for $l \leq 2$, but for $l \geq 3$, because of the great number of possible cases which we must consider, the completeness is not proved, and there is a possibility that some AC are not included into the catalogue.

In this catalogue for every AC is given a list of corresponding simple antisymmetry groups of the M^1-type, connections between AC in the case of transition from $m = 1$ to $m = 2$ and the table of numbers N_m and (N_{m-1}). The notation used and the method for obtaining results are the same as in the example for the **pm** symmetry group given in Section 3. In AC by parentheses ( ) is denoted the obligation of cyclic permutation of appertaining elements, by [ ] the obligation of simultaneous commutation of elements; the elements in / / parentheses remain fixed on their places. All AC obtained in previous studies of the theory of simple and multiple antisymmetry for $1 \leq l \leq 4$ [**1**, **4**, **5**, **6**, **7**, **8**, **11**] are included in this catalogue and denoted by $*$.

$\boxed{l = 1}$

1.1*	$\{A\}$	$\{e_1\}$		$N_1 = 1$

$\boxed{l = 2}$

2.1*	$\{A\}\{B\}$	$\{E\}\{e_1\}$	$\longrightarrow 2.1$	
		$\{e_1\}\{E\}$	$\longrightarrow 2.1$	$2.1 \longrightarrow 3(2.1)$
		$\{e_1\}\{e_1\}$	$\longrightarrow 2.1$	
2.2*	$\{A, B\}$	$\{E, e_1\}$	$\longrightarrow 2.1$	$2.2 \longrightarrow (2.1) + (2.2)$
		$\{e_1, e_1\}$	$\longrightarrow 2.2$	
2.3*	$\{A, B, AB\}$	$\{E, e_1, e_1\}$	$\longrightarrow 2.2$	$2.3 \longrightarrow (2.2)$

	N_1	N_2	(N_1)
2.1*	3	6	2
2.2*	2	3	1
2.3*	1	1	

$\boxed{l = 3}$

3.1*	$\{A\}\{B\}\{C\}$	$\{E\}\{E\}\{e_1\}$	$\longrightarrow 3.1$	
		$\{E\}\{e_1\}\{E\}$	$\longrightarrow 3.1$	
		$\{e_1\}\{E\}\{E\}$	$\longrightarrow 3.1$	
		$\{E\}\{e_1\}\{e_1\}$	$\longrightarrow 3.1$	$3.1 \longrightarrow 7(3.1)$
		$\{e_1\}\{E\}\{e_1\}$	$\longrightarrow 3.1$	
		$\{E\}\{e_1\}\{e_1\}$	$\longrightarrow 3.1$	
		$\{e_1\}\{e_1\}\{e_1\}$	$\longrightarrow 3.1$	

3.2*	$\{A,B\}\{C\}$	$\{E,e_1\}\{E\}$	$\longrightarrow 3.1$	
		$\{E,e_1\}\{e_1\}$	$\longrightarrow 3.1$	
		$\{E,E\}\{e_1\}$	$\longrightarrow 3.2$	$3.2 \longrightarrow 2(3.1)+3(3.2)$
		$\{e_1,e_1\}\{E\}$	$\longrightarrow 3.2$	
		$\{e_1,e_1\}\{e_1\}$	$\longrightarrow 3.2$	
3.3	(A,B,C,AB,AC,BC,ABC)			
	(E,E,e_1,e_1,E,e_1,e_1)		$\longrightarrow 3.1$	
	(E,e_1,E,e_1,E,e_1,e_1)		$\longrightarrow 3.1$	$3.3 \to 4(3.1)$
	(E,E,e_1,E,e_1,e_1,e_1)		$\longrightarrow 3.1$	
	(E,E,E,e_1,e_1,e_1,e_1)		$\longrightarrow 3.1$	
3.4*	$\{A,B\}\{C,ABC\}$			
	$\{E,e_1\}\{E,e_1\}$		$\longrightarrow 3.1$	
	$\{e_1,e_1\}\{E,E\}$		$\longrightarrow 3.4$	$3.4 \longrightarrow (3.1)+3(3.4)$
	$\{E,E\}\{e_1,e_1\}$		$\longrightarrow 3.4$	
	$\{e_1,e_1\}\{e_1,e_1\}$		$\longrightarrow 3.4$	
3.5*	(A,B,C)	(E,E,e_1)	$\longrightarrow 3.1$	
		(E,e_1,e_1)	$\longrightarrow 3.1$	$3.5 \longrightarrow 2(3.1)+(3.5)$
		(e_1,e_1,e_1)	$\longrightarrow 3.5$	
3.6	(A,B,C,ABC)			
	(E,E,e_1,e_1)		$\longrightarrow 3.1$	
	(E,e_1,E,e_1)		$\longrightarrow 3.2$	$3.6 \longrightarrow (3.1)+(3.2)+(3.6)$
	(e_1,e_1,e_1,e_1)		$\longrightarrow 3.6$	
3.7*	$\{A,B,C\}$	$\{E,E,e_1\}$	$\longrightarrow 3.2$	
		$\{E,e_1,e_1\}$	$\longrightarrow 3.2$	$3.7 \longrightarrow 2(3.2)+(3.7)$
		$\{e_1,e_1,e_1\}$	$\longrightarrow 3.7$	
3.8*	$\{\{A,B\},\{C,ABC\}\}$			
	$\{\{E,e_1\},\{E,e_1\}\}$		$\longrightarrow 3.2$	
	$\{\{E,E\},\{e_1,e_1\}\}$		$\longrightarrow 3.4$	$3.8 \longrightarrow (3.2)+(3.4)+(3.8)$
	$\{\{e_1,e_1\},\{e_1,e_1\}\}$		$\longrightarrow 3.8$	
3.9*	$\{A,B,C,ABC\}$			
	$\{E,E,e_1,e_1\}$		$\longrightarrow 3.4$	$3.9 \longrightarrow (3.4)+(3.9)$
	$\{e_1,e_1,e_1,e_1\}$		$\longrightarrow 3.9$	
3.10*	$\{A,B,C,AB,AC,BC,ABC\}$			
	$\{E,E,E,e_1,e_1,e_1,e_1\}$		$\longrightarrow 3.9$	$3.10 \longrightarrow (3.9)$

	N_1	N_2	N_3	(N_1)	(N_2)
3.1*	7	42	168	6	24
3.2*	5	24	84	4	12
3.3	4	24	96	3	15
3.4*	4	15	42	3	6
3.5*	3	14	56	2	8
3.6	3	12	42	2	6
3.7*	3	10	28	2	4
3.8*	3	9	21	2	3
3.9*	2	4	7	1	1
3.10*	1	1	1		

$\boxed{l = 4}$

4.1* $\{A\}\{B\}\{C\}\{D\}$

$\{E\}\{E\}\{E\}\{e_1\} \longrightarrow 4.1$
$\{E\}\{E\}\{e_1\}\{E\} \longrightarrow 4.1$
$\{E\}\{e_1\}\{E\}\{E\} \longrightarrow 4.1$
$\{e_1\}\{E\}\{E\}\{E\} \longrightarrow 4.1$
$\{E\}\{E\}\{e_1\}\{e_1\} \longrightarrow 4.1$
$\{E\}\{e_1\}\{E\}\{e_1\} \longrightarrow 4.1$
$\{e_1\}\{E\}\{E\}\{e_1\} \longrightarrow 4.1$
$\{E\}\{e_1\}\{e_1\}\{E\} \longrightarrow 4.1 \qquad 4.1 \longrightarrow 15(4.1)$
$\{e_1\}\{E\}\{e_1\}\{E\} \longrightarrow 4.1$
$\{e_1\}\{e_1\}\{E\}\{E\} \longrightarrow 4.1$
$\{E\}\{e_1\}\{e_1\}\{e_1\} \longrightarrow 4.1$
$\{e_1\}\{E\}\{e_1\}\{e_1\} \longrightarrow 4.1$
$\{e_1\}\{e_1\}\{E\}\{e_1\} \longrightarrow 4.1$
$\{e_1\}\{e_1\}\{e_1\}\{E\} \longrightarrow 4.1$
$\{e_1\}\{e_1\}\{e_1\}\{e_1\} \longrightarrow 4.1$

4.2* $\{A, B\}\{C\}\{D\}$

$\{E, e_1\}\{E\}\{E\} \longrightarrow 4.1$
$\{E, e_1\}\{e_1\}\{E\} \longrightarrow 4.1$
$\{E, e_1\}\{E\}\{e_1\} \longrightarrow 4.1$
$\{E, e_1\}\{e_1\}\{e_1\} \longrightarrow 4.1$
$\{E, E\}\{e_1\}\{E\} \longrightarrow 4.2$
$\{E, E\}\{E\}\{e_1\} \longrightarrow 4.2 \qquad 4.2 \longrightarrow 4(4.1) + 7(4.2)$
$\{e_1, e_1\}\{E\}\{E\} \longrightarrow 4.2$
$\{E, E\}\{e_1\}\{e_1\} \longrightarrow 4.2$
$\{e_1, e_1\}\{e_1\}\{E\} \longrightarrow 4.2$

$\{e_1, e_1\}\{E\}\{e_1\} \longrightarrow 4.2$
$\{e_1, e_1\}\{e_1\}\{e_1\} \longrightarrow 4.2$

4.3 $([A,B],[C,ABC],[D,ABD],[AC,BC],[AD,BD],$
$[CD,ABCD],[ACD,BCD])$
$([e_1,E],[E,e_1],[E,e_1],[e_1,E],[e_1,E],[E,e_1],[e_1,E]) \longrightarrow 4.1$
$([E,E],[e_1,e_1],[E,E],[e_1,e_1],[E,E],[e_1,e_1],[e_1,e_1]) \longrightarrow 4.1$
$([E,E],[E,E],[e_1,e_1],[E,E],[e_1,e_1],[e_1,e_1],[e_1,e_1]) \longrightarrow 4.1$
$([e_1,e_1],[E,E],[E,E],[e_1,e_1],[e_1,e_1],[E,E],[e_1,e_1]) \longrightarrow 4.1$
$([e_1,E],[E,e_1],[e_1,E],[e_1,E],[E,e_1],[e_1,E],[E,e_1]) \longrightarrow 4.1$
$([E,e_1],[E,e_1],[e_1,E],[E,e_1],[e_1,E],[e_1,E],[e_1,E]) \longrightarrow 4.1$
$([E,E],[e_1,e_1],[e_1,e_1],[e_1,e_1],[e_1,e_1],[E,E],[E,E]) \longrightarrow 4.1$
$([e_1,E],[e_1,E],[e_1,E],[E,e_1],[E,e_1],[E,e_1],[e_1,E]) \longrightarrow 4.1$
$([E,e_1],[E,e_1],[E,e_1],[E,e_1],[E,e_1],[E,e_1],[E,e_1]) \longrightarrow 4.3$
$4.3 \longrightarrow 8(4.1) + (4.3)$

4.4* $\{A,B\}\{C,D\}\{AC,BD\}$
$\{E,E\}\{E,e_1\}\{E,e_1\} \longrightarrow 4.1$
$\{E,e_1\}\{E,E\}\{E,e_1\} \longrightarrow 4.1$
$\{E,e_1\}\{e_1,e_1\}\{E,e_1\} \longrightarrow 4.1$
$\{E,e_1\}\{E,e_1\}\{E,E\} \longrightarrow 4.1$
$\{E,e_1\}\{E,e_1\}\{e_1,e_1\} \longrightarrow 4.1$ $\quad 4.4 \longrightarrow 6(4.1) + 3(4.4)$
$\{e_1,e_1\}\{E,e_1\}\{E,e_1\} \longrightarrow 4.1$
$\{e_1,e_1\}\{e_1,e_1\}\{E,E\} \longrightarrow 4.4$
$\{E,E\}\{e_1,e_1\}\{e_1,e_1\} \longrightarrow 4.4$
$\{e_1,e_1\}\{E,E\}\{e_1,e_1\} \longrightarrow 4.4$

4.5* $\{A\}\{B,C\}\{D,BCD\}$
$\{E\}\{E,e_1\}\{E,e_1\} \longrightarrow 4.1$
$\{e_1\}\{E,e_1\}\{E,e_1\} \longrightarrow 4.1$
$\{e_1\}\{e_1,e_1\}\{E,E\} \longrightarrow 4.5$
$\{E\}\{E,E\}\{e_1,e_1\} \longrightarrow 4.5$
$\{E\}\{e_1,e_1\}\{e_1,e_1\} \longrightarrow 4.5$ $\quad 4.5 \longrightarrow 2(4.5) + 7(4.5)$
$\{e_1\}\{e_1,e_1\}\{e_1,e_1\} \longrightarrow 4.5$
$\{e_1\}\{E,E\}\{e_1,e_1\} \longrightarrow 4.5$
$\{E\}\{e_1,e_1\}\{E,E\} \longrightarrow 4.5$
$\{e_1\}\{E,E\}\{E,E\} \longrightarrow 4.5$

4.6* $\{A,B\}\{C,D\}$
$\{E,e_1\}\{E,e_1\} \longrightarrow 4.1$
$\{E,e_1\}\{E,E\} \longrightarrow 4.2$
$\{E,E\}\{E,e_1\} \longrightarrow 4.2$
$\{e_1,e_1\}\{E,e_1\} \longrightarrow 4.2$ $\quad 4.6 \longrightarrow (4.1) + 4(4.2) + 3(4.6)$

	$\{E,e_1\}\{e_1,e_1\}$	$\longrightarrow 4.2$	
	$\{e_1,e_1\}\{E,E\}$	$\longrightarrow 4.6$	
	$\{E,E\}\{e_1,e_1\}$	$\longrightarrow 4.6$	
	$\{e_1,e_1\}\{e_1,e_1\}$	$\longrightarrow 4.6$	
4.7	$\{B,AB\}\{C,AC\}\{D,AD\}$		
	$\{E,e_1\}\{E,e_1\}\{E,e_1\}$	$\longrightarrow 4.1$	
	$\{E,E\}\{E,E\}\{e_1,e_1\}$	$\longrightarrow 4.7$	
	$\{E,E\}\{e_1,e_1\}\{E,E\}$	$\longrightarrow 4.7$	
	$\{E,E\}\{e_1,e_1\}\{e_1,e_1\}$	$\longrightarrow 4.7$	$4.7 \longrightarrow (4.1)+7(4.7)$
	$\{e_1,e_1\}\{E,E\}\{e_1,e_1\}$	$\longrightarrow 4.7$	
	$\{e_1,e_1\}\{e_1,e_1\}\{E,E\}$	$\longrightarrow 4.7$	
	$\{e_1,e_1\}\{E,E\}\{E,E\}$	$\longrightarrow 4.7$	
	$\{e_1,e_1\}\{e_1,e_1\}\{e_1,e_1\}$	$\longrightarrow 4.7$	
4.8	$\{A\}(B,C,D)$		
	$\{E\}(e_1,E,E)$	$\longrightarrow 4.1$	
	$\{e_1\}(e_1,E,E)$	$\longrightarrow 4.1$	
	$\{E\}(e_1,e_1,E)$	$\longrightarrow 4.1$	
	$\{e_1\}(e_1,e_1,E)$	$\longrightarrow 4.1$	$4.8 \longrightarrow 4(4.1)+3(4.8)$
	$\{e_1\}(E,E,E)$	$\longrightarrow 4.8$	
	$\{E\}(e_1,e_1,e_1)$	$\longrightarrow 4.8$	
	$\{e_1\}(e_1,e_1,e_1)$	$\longrightarrow 4.8$	
4.9	$(/A,B/,/C,ABC/,/D,ABD/,/ACD,BCD/)$		
	$(/e_1,E/,/E,e_1/,/E,e_1/,/e_1,E/)$	$\longrightarrow 4.1$	
	$(/e_1,E/,/E,e_1/,/e_1,E/,/E,e_1/)$	$\longrightarrow 4.1$	
	$(/E,E/,/e_1,e_1/,/E,E/,/e_1,e_1/)$	$\longrightarrow 4.2$	
	$(/E,E/,/E,E/,/e_1,e_1/,/e_1,e_1/)$	$\longrightarrow 4.2$	
	$(/E,e_1/,/E,e_1/,/E,e_1/,/E,e_1/)$	$\longrightarrow 4.9$	
	$(/e_1,E/,/e_1,E/,/e_1,E/,/e_1,E/)$	$\longrightarrow 4.9$	
	$(/e_1,e_1/,/e_1,e_1/,/e_1,e_1/,/e_1,e_1/)$	$\longrightarrow 4.9$	
	$4.9 \longrightarrow 2(4.1)+2(4.2)+3(4.9)$		
4.10*	$\{A,B,C\}\{D\}$		
	$\{E,E,e_1\}\{E\}$	$\longrightarrow 4.2$	
	$\{E,E,e_1\}\{e_1\}$	$\longrightarrow 4.2$	
	$\{E,e_1,e_1\}\{E\}$	$\longrightarrow 4.2$	
	$\{E,e_1,e_1\}\{e_1\}$	$\longrightarrow 4.2$	$4.10 \longrightarrow 4(4.2)+3(4.10)$
	$\{e_1,e_1,e_1\}\{E\}$	$\longrightarrow 4.10$	
	$\{e_1,e_1,e_1\}\{e_1\}$	$\longrightarrow 4.10$	
	$\{E,E,E\}\{e_1\}$	$\longrightarrow 4.10$	

4.11* $\{\{A,B\},\{CA,CB\}\}\{D,CD\}$

$\{\{E,E\},\{e_1,e_1\}\}\{E,e_1\} \longrightarrow 4.2$

$\{\{E,e_1\},\{E,e_1\}\}\{E,e_1\} \longrightarrow 4.2$

$\{\{E,e_1\},\{E,e_1\}\}\{E,e_1\} \longrightarrow 4.5$

$\{\{E,e_1\},\{E,e_1\}\}\{e_1,e_1\} \longrightarrow 4.5$

$\{\{E,E\},\{E,E\}\}\{e_1,e_1\} \longrightarrow 4.11$

$\{\{e_1,e_1\},\{e_1,e_1\}\}\{E,E\} \longrightarrow 4.11$

$\{\{e_1,e_1\},\{e_1,e_1\}\}\{e_1,e_1\} \longrightarrow 4.11$

$4.11 \longrightarrow 2(4.2) + 2(4.5) + 3(4.11)$

4.12* $\{[A,B],[C,D]\}$

$\{[e_1,E],[E,E]\} \longrightarrow 4.1$

$\{[e_1,e_1],[E,e_1]\} \longrightarrow 4.1$

$\{[e_1,e_1],[E,E]\} \longrightarrow 4.4$

$\{[e_1,E],[e_1,E]\} \longrightarrow 4.4$

$\{[e_1,E],[E,e_1]\} \longrightarrow 4.4$

$\{[e_1,e_1],[e_1,e_1]\} \longrightarrow 4.12$

$4.12 \longrightarrow 2(4.1) + 3(4.4) + (4.12)$

4.13 $\{\{B,AB\},\{C,AC\}\}\{D,AD\}$

$\{\{E,e_1\},\{E,e_1\}\}\{E,e_1\} \longrightarrow 4.2$

$\{\{E,E\},\{e_1,e_1\}\}\{E,E\} \longrightarrow 4.7$

$\{\{E,E\},\{e_1,e_1\}\}\{e_1,e_1\} \longrightarrow 4.7$

$\{\{E,E\},\{E,E\}\}\{e_1,e_1\} \longrightarrow 4.13$

$\{\{e_1,e_1\},\{e_1,e_1\}\}\{E,E\} \longrightarrow 4.13$

$\{\{e_1,e_1\},\{e_1,e_1\}\}\{e_1,e_1\} \longrightarrow 4.13$

$4.13 \longrightarrow (4.2) + 2(4.7) + 3(4.13)$

4.14 (A,B,C,D)

$(e_1,E,E,E) \longrightarrow 4.1$

$(e_1,e_1,E,E) \longrightarrow 4.1$

$(e_1,e_1,e_1,E) \longrightarrow 4.1$

$(e_1,E,e_1,E) \longrightarrow 4.4$

$(e_1,e_1,e_1,e_1) \longrightarrow 4.14$

$4.14 \longrightarrow 3(4.1) + (4.4) + (4.14)$

4.15* $(C,A,AC)\{(B,C,ABC),(BD,BCD,ABCD)\}$

$(E,e_1,e_1)\{(E,E,e_1),(e_1,e_1,E)\} \longrightarrow 4.1$

$(E,e_1,e_1)\{(E,E,e_1),(E,E,e_1)\} \longrightarrow 4.2$

$(E,e_1,e_1)\{(E,e_1,e_1),(E,e_1,e_1)\} \longrightarrow 4.2$

$(E,E,E)\{(E,E,E),(e_1,e_1,e_1)\} \longrightarrow 4.8$

$(E,E,E)\{(e_1,e_1,e_1),(e_1,e_1,e_1)\} \longrightarrow 4.15$

$4.15 \longrightarrow (4.1) + 2(4.2) + (4.8) + (4.15)$

4.16*
$\{\{A,B\},\{C,D\}\}$
$\{\{E,e_1\},\{E,E\}\} \longrightarrow 4.2$
$\{\{E,e_1\},\{e_1,e_1\}\} \longrightarrow 4.2$
$\{\{E,e_1\},\{E,e_1\}\} \longrightarrow 4.4$
$\{\{E,E\},\{e_1,e_1\}\} \longrightarrow 4.6$
$\{\{e_1,e_1\},\{e_1,e_1\}\} \longrightarrow 4.16$
$4.16 \longrightarrow 2(4.2)+(4.4)+(4.6)+(4.16)$

4.17
$(\{A,B\},\{C,ABC\},\{D,ABD\},\{AC,BC\},\{AD,BD\},$
$\{CD,ABCD\},\{ACD,BCD\})$
$(\{E,e_1\},\{E,e_1\},\{E,e_1\},\{E,e_1\},\{E,e_1\},\{E,e_1\},\{E,e_1\}) \longrightarrow 4.3$
$(\{E,E\},\{e_1,e_1\},\{E,E\},\{e_1,e_1\},\{E,E\},\{e_1,e_1\},\{e_1,e_1\}) \longrightarrow 4.7$
$(\{E,E\},\{E,E\},\{e_1,e_1\},\{E,E\},\{e_1,e_1\},\{e_1,e_1\},\{e_1,e_1\}) \longrightarrow 4.7$
$(\{e_1,e_1\},\{E,E\},\{E,E\},\{e_1,e_1\},\{e_1,e_1\},\{E,E\},\{e_1,e_1\}) \longrightarrow 4.7$
$(\{E,E\},\{e_1,e_1\},\{e_1,e_1\},\{e_1,e_1\},\{e_1,e_1\},\{E,E\},\{E,E\}) \longrightarrow 4.7$
$4.17 \longrightarrow (4.3)+4(4.7)$

4.18*
$\{A,B,AB\}\{C,D\}$
$\{E,e_1,e_1\}\{E,e_1\} \longrightarrow 4.2$
$\{E,e_1,e_1\}\{E,E\} \longrightarrow 4.6$
$\{E,e_1,e_1\}\{e_1,e_1\} \longrightarrow 4.6$
$\{E,E,E\}\{E,e_1\} \longrightarrow 4.10$
$\{E,E,E\}\{e_1,e_1\} \longrightarrow 4.18$
$4.18 \longrightarrow (4.2)+2(4.6)+(4.10)+(4.18)$

4.19*
$\{A,B,C,ABC\}\{D\}$
$\{E,E,e_1,e_1\}\{E\} \longrightarrow 4.5$
$\{E,E,e_1,e_1\}\{e_1\} \longrightarrow 4.5$
$\{e_1,e_1,e_1,e_1\}\{E\} \longrightarrow 4.19$
$\{e_1,e_1,e_1,e_1\}\{e_1\} \longrightarrow 4.19$
$\{E,E,E,E\}\{e_1\} \longrightarrow 4.19$
$4.19 \longrightarrow 2(4.5)+3(4.19)$

4.20
$\{\{A,B\},\{C,ABC\}\}\{\{D,ABD\},\{ACD,BCD\}\}$
$\{\{E,e_1\},\{E,e_1\}\}\{\{E,e_1\},\{E,e_1\}\} \longrightarrow 4.5$
$\{\{E,E\},\{e_1,e_1\}\}\{\{E,E\},\{e_1,e_1\}\} \longrightarrow 4.7$
$\{\{E,E\},\{E,E\}\}\{\{e_1,e_1\},\{e_1,e_1\}\} \longrightarrow 4.20$
$\{\{e_1,e_1\},\{e_1,e_1\}\}\{\{E,E\},\{E,E\}\} \longrightarrow 4.20$
$\{\{e_1,e_1\},\{e_1,e_1\}\}\{\{e_1,e_1\},\{e_1,e_1\}\} \longrightarrow 4.20$
$4.20 \longrightarrow (4.5)+(4.7)+3(4.20)$

4.21
$(\{A,AD\},\{B,BD\},\{C,CD\})$
$(\{E,E\},\{e_1,e_1\},\{e_1,e_1\}) \longrightarrow 4.7$

$(\{E,E\},\{E,E\},\{e_1,e_1\}) \longrightarrow 4.7$
$(\{E,e_1\},\{E,e_1\},\{E,e_1\}) \longrightarrow 4.8$
$(\{e_1,e_1\},\{e_1,e_1\},\{e_1,e_1\}) \longrightarrow 4.21$
$4.21 \longrightarrow 2(4.7) + (4.8) + (4.21)$

4.22 $\{A,B,C,D\}$
$\{E,E,e_1,e_1\} \longrightarrow 4.6$
$\{E,E,E,e_1\} \longrightarrow 4.10$
$\{E,e_1,e_1,e_1\} \longrightarrow 4.10$
$\{e_1,e_1,e_1,e_1\} \longrightarrow 4.22$
$4.22 \longrightarrow (4.6) + 2(4.10) + (4.22)$

4.23 $(\{A,B\},\{C,ABC\},\{D,ABD\},\{ACD,BCD\})$
$(\{E,E\},\{E,E\},\{e_1,e_1\},\{e_1,e_1\}) \longrightarrow 4.7$
$(\{E,e_1\},\{E,e_1\},\{E,e_1\},\{E,e_1\}) \longrightarrow 4.9$
$(\{E,E\},\{e_1,e_1\},\{E,E\},\{e_1,e_1\}) \longrightarrow 4.13$
$(\{e_1,e_1\},\{e_1,e_1\},\{e_1,e_1\},\{e_1,e_1\}) \longrightarrow 4.23$
$4.23 \longrightarrow (4.7) + (4.9) + (4.13) + (4.23)$

4.24* $\{\{B,AC\},\{C,AC\},\{D,AD\}\}$
$\{\{E,e_1\},\{E,e_1\},\{E,e_1\}\} \longrightarrow 4.11$
$\{\{E,E\},\{E,E\},\{e_1,e_1\}\} \longrightarrow 4.13$
$\{\{e_1,e_1\},\{E,E\},\{e_1,e_1\}\} \longrightarrow 4.13$
$\{\{e_1,e_1\},\{e_1,e_1\},\{e_1,e_1\}\} \longrightarrow 4.24$
$4.24 \longrightarrow (4.11) + 2(4.13) + (4.24)$

4.25 $\{\{\{A,B\},\{C,ABC\}\},\{\{D,ABD\},\{ACD,BCD\}\}\}$
$\{\{\{E,e_1\},\{E,e_1\}\},\{\{E,e_1\},\{E,e_1\}\}\} \longrightarrow 4.11$
$\{\{\{E,E\},\{e_1,e_1\}\},\{\{E,E\},\{e_1,e_1\}\}\} \longrightarrow 4.13$
$\{\{\{E,E\},\{E,E\}\},\{\{e_1,e_1\},\{e_1,e_1\}\}\} \longrightarrow 4.20$
$\{\{\{e_1,e_1\},\{e_1,e_1\}\},\{\{e_1,e_1\},\{e_1,e_1\}\}\} \longrightarrow 4.25$
$4.25 \longrightarrow (4.11) + (4.13) + (4.20) + (4.25)$

4.26 $\{A,B,C,ABC\}\{D,ABD,ACD,BCD\}$
$\{E,E,e_1,e_1\}\{E,E,e_1,e_1\} \longrightarrow 4.7$
$\{E,E,E,E\}\{e_1,e_1,e_1,e_1\} \longrightarrow 4.26$
$\{e_1,e_1,e_1,e_1\}\{E,E,E,E\} \longrightarrow 4.26$
$\{e_1,e_1,e_1,e_1\}\{e_1,e_1,e_1,e_1\} \longrightarrow 4.26$
$4.26 \longrightarrow (4.7) + 3(4.26)$

4.27* $\{\{A,B\},\{C,D\},\{AC,BD\}\}$
$\{\{E,E\},\{E,e_1\},\{E,e_1\}\} \longrightarrow 4.4$
$\{\{E,e_1\},\{E,e_1\},\{e_1,e_1\}\} \longrightarrow 4.4$

$\{\{E,E\},\{e_1,e_1\},\{e_1,e_1\}\} \longrightarrow 4.12$

$4.27 \longrightarrow 2(4.4)+(4.12)$

4.28 $\{\{A,B\},\{C,ABC\},\{D,ABD\},\{ACD,BCD\}\}$

$\{\{E,E\},\{E,E\},\{e_1,e_1\},\{e_1,e_1\}\} \longrightarrow 4.19$

$\{\{E,e_1\},\{E,e_1\},\{E,e_1\},\{E,e_1\}\} \longrightarrow 4.20$

$\{\{e_1,e_1\},\{e_1,e_1\},\{e_1,e_1\},\{e_1,e_1\}\} \longrightarrow 4.28$

$4.28 \longrightarrow (4.19)+(4.20)+(4.28)$

4.29* $\{A,B,C,D,ABC,ABD,ACD,BCD\}$

$\{E,E,E,E,e_1,e_1,e_1,e_1\} \longrightarrow 4.26$

$\{e_1,e_1,e_1,e_1,e_1,e_1,e_1,e_1\} \longrightarrow 4.29$

$4.29 \longrightarrow (4.26)+(4.29)$

4.30 $\{A,B,C,D,AB,AC,AD,BC,BD,CD,ABC,ABD,ACD,BCD,ABCD\}$

$\{E,E,E,E,E,E,E,e_1,e_1,e_1,e_1,e_1,e_1,e_1,e_1\} \longrightarrow 4.24$

$4.29 \longrightarrow (4.24)$

	N_1	N_2	N_3	N_4	(N_1)	(N_2)	(N_3)
4.1*	15	210	2520	20160	14	168	1344
4.2*	11	126	1344	10080	10	96	672
4.3	9	120	1440	11520	8	96	768
4.4*	9	108	1260	10080	8	84	672
4.5*	9	84	756	5040	8	60	336
4.6*	8	75	714	5040	7	54	336
4.7	8	63	462	2520	7	42	168
4.8	7	74	840	6720	6	56	448
4.9	7	66	672	5040	6	48	336
4.10*	7	58	504	5040	6	40	224
4.11*	7	54	420	2520	6	36	168
4.12*	6	57	630	5040	5	42	336
4.13	6	39	252	1260	5	24	84
4.14	5	54	630	5040	4	42	336
4.15*	5	44	448	3360	4	32	224
4.16*	5	39	357	2520	4	27	168
4.17	5	36	264	1440	4	24	96
4.18*	5	34	266	1680	4	22	112
4.19*	5	28	168	840	4	16	56
4.20	5	27	147	630	4	15	42
4.21	4	23	154	840	3	14	56
4.22	4	22	147	840	3	13	56
4.23	4	21	126	630	3	12	42
4.24*	4	19	98	420	3	10	28
4.25	4	18	84	315	3	9	21
4.26	4	16	63	210	3	7	14

4.27*	3	21	210	1680	2	15	105
4.28	3	10	35	105	2	4	7
4.29*	2	4	8	15	1	1	1
4.30	1	1	1	1			

Besides all AC found in practice during previous studies of the theory of simple and multiple antisymmetry for $1 \leq l \leq 4$ in this catalogue there are some AC which are not found before.

Conjecture 1: Every abstract algebraic term formed in accordance with Definition 1 is AC of some symmetry group.

Most of the AC given in this catalogue, which are not found in earlier practice, satisfy Conjecture 1. For example, AC 4.22 corresponds to the symmetry group **mmmm** of the category G_{40}, and AC 4.30 corresponds to the symmetry group **P1111** of the category G_4.

If Conjecture 1 is valid, AC 4.21 and 4.22 are counter-examples of the supposition [**1**, p. 138] that equality of the first and last members of the series $N_m(G)$ and $N_m(G')$ implies equality of the second members of these series.

Besides the numbers N_m in the catalogue of non-isomorphic AC $(1 \leq l \leq 4)$ are given the numbers (N_{m-1}), discussed in Section 2 in the case of (p, l)-symmetry at p even. The meaning of these numbers in the case of multiple antisymmetry is absolutely clear from the identity of (l)- and $(1, l-1)$-multiple antisymmetry. Consequently, the relation

$$(N_m) = N_m - (2^m - 1)(N_{m-1}), \qquad (N_0) = 1, \quad 1 \leq m \leq l,$$

holds.

Conjecture 2: Every series N_m obtained from AC_l formed by l generators is identical with some series (N_{m-1}) obtained from corresponding AC_{l+1} formed by $l+1$ generators.

As the examples of AC_{l+1} and AC_l which satisfy the Conjecture 2 for $1 \leq l \leq 4$, it is possible to use the pairs of AC: 2.2 and 1.1, 3.4 and 2.1, 3.8 and 2.2, 3.9 and 2.3, 4.7 and 3.1, 4.13 and 3.2, 4.17 and 3.3, 4.20 and 3.4, 4.21 and 3.5, 4.23 and 3.6, 4.24 and 3.7, 4.25 and 3.8, 4.28 and 3.9, 4.19 and 3.10.

Conjecture 3: Let AC_l formed by generators $A_1, \ldots, A_l$ be given. Then by substitution $A_i' = A_i A_{i+1}$, $i = \overline{1, l}$, can be obtained a new AC_{l+1} such that AC_l and AC_{l+1} satisfy Conjecture 2.

The discussion about numbers (N_{m-1}) can be generalised and extended to numbers (N_{m-k}), which play an important role in the derivation of junior groups of (P, l)-symmetry. For example, the numbers (N_{m-2}) appear in the case of junior groups of complete $(p2, l)$-symmetry for p even. In the theory of multiple antisymmetry the meaning of the numbers (N_{m-k}) can be seen from the identity of (l)- and $(k, l-k)$-mulitiple antisymmetry.

The study of particular non-isomorphic AC for $l > 4$ is almost a technical problem. However, a proof of completeness of the catalogue of non-isomorphic AC at $l > 2$ is immensely important and one on the aims of future studies of the theory of simple and multiple antisymmetry must be the construction of an algorythm, whick makes possible direct obtaining of all non-isomorphic AC formed by l generators.

In many cases, especially for AC with a large number of generators, for the computing of numbers N_m it is possible to use the direct product of AC.

5. The direct product of *AC*. *Definition* 2. Let AC' and AC'' with disjoint sets of generators be given. The new $AC = AC'AC''$ obtained by concatenating AC' and AC'' is called the direct product of AC' and AC''.

THEOREM 3. *Let N_m, N'_m, N''_m be the series of numbers defined by AC, AC', AC'' respectively. Then the relation*

$$N_m = \sum_{\substack{k+l\geq m\\ m\geq k,\, l\geq 0}} 2^{(m-k)(m-l)} C(m, m-k, m-l) N'_k N''_m$$

holds, where

$$C(l,k,m) = \frac{(2^l-1)(2^{l-1}-1)\cdot\ldots\cdot(2^{l-k-m+1}-1)}{(2^k-1)(2^{k-1}-1)\cdot\ldots\cdot(2-1)(2^m-1)(2^{m-1}-1)\cdot\ldots\cdot(2-1)}.$$

By use of the values of $C(l,m,k)$ computed in [**1**, p. 86], we have for $m \leq 6$ the following relations

$$N_1 = N'_1N''_1 + N'_1 + N''_1,$$

$$N_2 = N'_2N''_2 + N'_2 + N''_2 + 3N'_1N''_2 + 3N'_2N'_1 + 6N'_1N''_1,$$

$$N_3 = N'_3N''_3 + N'_3 + N''_3 + 7N'_1N''_3 + 7N'_3N''_1 + 28N'_1N''_2 + 28N'_2N''_1 + 42N'_2N''_2$$

$$\begin{aligned} N_4 = {} & N'_4N''_4 + N'_4 + N''_4 + 15N'_1N''_4 + 15N'_4N''_1 + 15N'_3N''_4 + 15N'_4N''_3 \\ & + 35N'_2N''_4 + 35N'_4N''_2 + 120N'_1N''_3 + 120N'_3N''_1 + 210N'_3N''_3 \\ & + 420N'_2N''_3 + 420N'_3N''_2 + 560N'_2N''_2, \end{aligned}$$

$$\begin{aligned} N_5 = {} & N'_5N''_5 + N'_5 + N''_5 + 31N'_1N''_5 + 31N'_5N''_1 + 31N'_4N''_5 + 31N'_5N''_4 \\ & + 155N'_2N''_5 + 155N'_5N''_2 + 155N'_3N''_5 + 155N'_5N''_3 + 496N'_1N''_4 \\ & + 496N'_4N''_1 + 930N'_4N''_4 + 3720N'_2N''_4 + 3720N'_4N''_2 + 4340N'_3N''_4 \\ & + 4340N'_4N''_3 + 9920N'_2N''_3 + 9920N'_3N''_2 + 17360N'_3N''_3, \end{aligned}$$

$$\begin{aligned} N_6 = {} & N'_6N''_6 + N'_6 + N''_6 + 63N'_1N''_6 + 63N'_6N''_1 + 63N'_5N''_6 + 63N'_6N''_5 \\ & + 651N'_2N''_6 + 651N'_6N''_2 + 651N'_4N''_6 + 651N'_6N''_4 + 1395N'_3N''_6 \\ & + 1395N'_6N''_3 + 2016N'_1N''_5 + 2016N'_5N''_1 + 3906N'_5N''_5 + 31248N'_2N''_5 \end{aligned}$$

$$+ 31248N_5'N_2'' + 39060N_4'N_5'' + 39060N_5'N_4'' + 78120N_3'N_5''$$
$$+ 78120N_5'N_3'' + 166656N_2'N_4'' + 166656N_4'N_2'' + 312480N_3'N_4''$$
$$+ 312480N_4'N_3'' + 364560N_4'N_4'' + 714240N_3'N_3''.$$

As an illustration of the $AC = AC'AC''$ which satisfy Theorem 3, we give the following example

$$AC' = 2.2 = \{A, B\} \qquad N_1(2.2) = 2 \qquad N_2(2.2) = 3$$
$$AC'' = 2.1 = \{C\}\{D\} \qquad N_1(2.1) = 3 \qquad N_2(2.1) = 6$$
$$AC = \{A, B\}\{C\}\{D\} = 4.2$$

In accordance with Theorem 3,

$$N_1(4.2) = 2 \cdot 3 + 2 + 3 = 11$$
$$N_2(4.2) = 3 \cdot 6 + 3 + 6 + 3 \cdot 2 \cdot 6 + 3 \cdot 3 \cdot 3 + 6 \cdot 2 \cdot 3 = 126$$
$$N_3(4.2) = 28 \cdot 2 \cdot 6 + 28 \cdot 3 \cdot 3 + 42 \cdot 3 \cdot 6 = 1344$$
$$N_4(4.2) = 560 \cdot 3 \cdot 6 = 10080.$$

Other examples of $AC = AC'AC''$ from the catalogue of non-isomorphic AC at $1 \leq l \leq 4$ are $2.1 = (1.1)(1.1)$, $3.1 = (2.1)(1.1)$, $3.2 = (2.2)(1.1)$, $4.1 = (3.1)(1.1) = (2.1)(2.1)$, $4.2 = (3.2)(1.1)$, $4.6 = (2.2)(2.2)$, $4.8 = (3.5) \cdot (1.1)$, $4.10 = (3.7)(1.1)$, $4.18 = (2.3)(2.2)$, $4.19 = (3.9)(1.1)$.

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