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MODAL TRANSLATIONS OF HEYTING AND PEANO ARITHMETIC

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Abstract. First-order Heyting arithmetic is embedded by various modal translations in modal extensions of first-order Peano arithmetic which are included in Peano S4. Peano arithmetic is embedded by analogous modal translations in an S5-like extension of Heyting arithmetic. This last system is included in the modal extension of Heyting arithmetic where the necessity operator is equivalent to double negation and where Peano arithmetic can be embedded by a modal translation which amounts to a usual double-negation translation.

To embed a system S_1 in a system S_2 by a translation, i.e. one-one mapping, t from the language of S_1 into the language of S_2 means to show that for every formula A of the language of S_1 we have that A is provable in S_1 iff $t(A)$ is provable in S_2. It has been known for a long time that Heyting first-order logic H can be embedded in the first-order modal logic S4 by modal translations, i.e. translations which prefix the necessity operator $\Box$ to certain subformulae. However, it was only in the last decade that it was shown that by analogous translations one can embed first-order Heyting arithmetic HA in first-order Peano arithmetic based on S4, which we shall denote by PAS4 (see [**12**]). An elegant proof of this last embedding may be found in [**9**].

When one considers the embedding of H in first-order modal logics, only S4 is usually mentioned, and this might leave the wrong impression that only S4 does the job. As a matter of fact, one can embed H in other first-order modal logics in the neighbourhood of S4 which are based on normal propositional modal logics (a propositional modal logic is normal iff it includes the classical propositional calculus and all instances of the schema $\Box(A \to B) \to (\Box A \to \Box B)$, and is closed under *modus ponens*, substitution for propositional variables and necessitation, i.e. the rule *from* A *infer* $\Box A$. The class of normal propositional modal logics in which the

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Heyting propositional calculus can be embedded is investigated in [**8**] and [**7**]. In [**6**] and [**7**] the embedding of H in first-order modal systems different from S4 is considered too.

It is natural to ask whether HA can be embedded in first-order Peano arithmetic based on the other modal logics in the neighbourhood of S4, and not only in PAS4. We shall show here that this question can easily be answered affirmatively for some modal extensions of Peano arithmetic included in PAS4.

In [**15**] a considerable philosophical significance is given to S4 modal principles as a clue to an intuitionistic notion of provability. (In [**15**], and in papers referring to [**15**], systems based on S4 are called "epistemic", thus replacing a denomination rather well established among logicians by a term which seems to mean various things in philosophical logic.) The results which we shall present show that S4 is not sacrosanct and that modal notions which can help us to translate intuitionistic notions in classical terms need not be exactly those of S4.

This is what we mean to accomplish in the first part of this paper. In the second part we consider the converse modal translations from nonmodal systems based on classical logic into modal systems based on H. We show that classical first-order Peano arithmetic PA can be embedded by a modal translation into an S5-like extension of first-order Heyting arithmetic which we shall call HA5. Again, HA5 is not the only modal system based on HA for which we have such an embedding. Among the systems in the neighbourhood of HA5 in which we can embed PA we find in particular a system based on HA where $\Box A \leftrightarrow \neg\neg A$ holds. The embedding of PA by a modal translation into this last system will be essentially the same as the embedding of PA into HA by a double-negation translation, i.e. a translation which prefixes $\neg\neg$ to certain subformulae.

The results which we shall present are based essentially on the underlying logics and in principle do not involve any purely arithmetical properties of HA and PA. So, we may reasonably expect that these results could be extended to some other systems based on Heyting and classical logic. However, for the sake of definiteness we have preferred to stick to the particular systems HA and PA.

1. Modal translations of Heyting arithmetic

Let L be the language of first-order arithmetic with the logical constants $\rightarrow$, $\wedge$, $\vee$, $\neg$, $\forall$, $\exists$ and $=$, the nonlogical constants 0, $'$, $+$ and $\cdot$, and denumerably many individual variables, for which we use the schematic letters $x, y, z, \dots, x_1, \dots$. As schematic letters for terms of L we use $t, t_1, \dots$, and as schematic letters for formulae of L we use $A, B, C, \dots, A_1, \dots$. A schema of the form A^x_t will stand for the formula obtained from A by substituting t for every free occurrence of x provided the usual proviso for substitution is satisfied. As usual, $A \leftrightarrow B$ is defined as $(A \rightarrow B) \wedge (B \rightarrow A)$. The language $L\Box$ is L extended with the unary propositional operator $\Box$. For $L\Box$ we use the same schematic letters as for L.

The system HA in L has the following axiom-schemata and rules:

(I) $(A \to (B \to C)) \to ((A \to B) \to (A \to C))$, $A \to (B \to A)$,
$A \to (B \to (A \wedge B))$, $(A \wedge B) \to A$, $(A \wedge B) \to B$,
$A \to (A \vee B)$, $B \to (A \vee B)$, $(A \to C) \to ((B \to C) \to ((A \vee B) \to C))$,
$(A \to \neg B) \to (B \to \neg A)$, $\neg A \to (A \to B)$, $\dfrac{A \quad A \to B}{B}$,

(II) $\forall x\, A \to A^x_t$, $A^x_t \to \exists x\, A$, $x = x$, $x = y \to (A^z_x \to A^z_y)$, $\dfrac{B \to A}{B \to \forall x\, A}$, $\dfrac{A \to B}{\exists x\, A \to B}$,
provided x is not free in B in these two rules,

(III) $\neg x' = 0$, $x' = y' \to x = y$, $\forall x\,(A \to A^x_{x'}) \to (A^x_0 \to A)$,
$x + 0 = x$, $x + y' = (x + y)'$, $x \cdot 0 = 0$, $x \cdot y' = (x \cdot y) + x$.

The system PA in L is HA extended with $A \vee (A \to B)$.

Consider now the following modal axiom-schemata and rules for systems in the language $L\Box$:

($\Box$1) $\Box(A \to B) \to (\Box A \to \Box B)$,

($\Box$2) $\dfrac{A}{\Box A}$,

($\Box$3) $\Box A \to \Box\Box A$,

($\Box$4) $\Box\Box A \to \Box A$,

($\Box$5) $\neg\Box\neg(A \to A)$,

($\Box$6) $\Box(\Box A \vee \Box B) \to (\Box A \vee \Box B)$,

($\Box$7) $\Box\exists x\Box A \to \exists x\Box A$,

($\Box$8) $\Box A \to A$,

($\Box$9) $\Box(\Box(A \to \Box A) \to A) \to A$.

We shall introduce a number of modal systems in $L\Box$ obtained by extending the axiom-schemata and rules of HA and PA (now understood as axiom-schemata and rules for $L\Box$) with some of these modal postulates:

HA4 = HA + $\Box$1, $\Box$2, $\Box$3, $\Box$4, $\Box$5;

HA4N = HA4 + $\Box$6, $\Box$7 ($\Box$4 is superfluous in HA4N);

HAS4 = HA4N + $\Box$8 ($\Box$4–$\Box$7 are superfluous in HAS4);

HAGrz = HAS4 + $\Box$9 ($\Box$4–$\Box$8 are superfluous in HAGrz).

The systems PA4, PA4N, PAS4 and PAGrz are obtained from the respective HA systems by adding $A \vee (A \to B)$. In PAGrz the schema $\Box$3 is superfluous, as well as $\Box$4–$\Box$8 (see [**1**]). A variant of PAGrz was recently considered in [**10**].

It is easy to show that all these modal systems in $L\Box$ based on HA or PA are conservative extensions of respectively HA or PA in L. In a proof of a formula A of L in one of these modal systems just delete every $\Box$; the result is a proof of A in HA or PA.

The modal translation m is the one-one mapping from L into $L\Box$ defined by the following recursive clauses:

$m(t_1 = t_2)$ is $\Box(t_1 = t_2)$,

$m(A\ \alpha\ B)$ is $\Box(m(A)\ \alpha\ m(B))$, where α is $\to$, $\wedge$ or $\vee$,

$m(\beta A)$ is $\Box\beta m(A)$, where β is $\neg$, $\forall x$ or $\exists x$.

In other words, in $m(A)$ the necessity operator $\Box$ is prefixed to every subformula of A.

By an easy induction on the lenght of proof of A we can establish the following lemma:

LEMMA 1. *If* A *is provable in* HA*, then* $m(A)$ *is provable in* HA4*.*

Since HA4 and all its extensions are closed under replacement of equivalents, and since $\Box(\Box B \wedge \Box C) \leftrightarrow (\Box B \wedge \Box C)$ is provable in HA4, the formula obtained from $m(A)$ by omitting $\Box$ in front of conjunctions will be equivalent in HA4 to $m(A)$. This would enable us to achieve a certain economy in our translation.

Consider now the translation m' from L into $L\Box$ where $m'(A)$ is obtained by prefixing $\Box$ to every *proper* subformula of A, i.e. by omitting from $m(A)$ the main necessity operator. We can prove the following lemma:

LEMMA 2. *If* A *is provable in* HA*, then* $m'(A)$ *is provable in* HA4*.*

Proof. In HA4 we can prove:

$$\Box(\Box B \rightarrow \Box C) \rightarrow (\Box B \rightarrow \Box C),$$
$$\Box(\Box B \wedge \Box C) \rightarrow (\Box B \wedge \Box C),$$
$$\Box\neg\Box B \rightarrow \neg\Box B,$$
$$\Box\forall x\Box B \rightarrow \forall x\Box B.$$

Now suppose $\Box(t_1 = t_2)$ is provable in HA4. By deleting every $\Box$ in the proof of $\Box(t_1 = t_2)$ we obtain a proof of $t_1 = t_2$ in HA, and hence in HA4. Suppose our theorem A of HA is of the form $B \vee C$. Then either B or C is provable in HA. Let this be B (with C we proceed analogously). Then $m(B)$ is provable in HA4, and hence $m(B) \vee m(C)$, i.e. $m'(A)$, is too. Suppose our theorem A of HA is of the form $\exists x\, B$. Then for some term t (indeed, for some numeral t) we have that B^x_t is provable in HA, and hence $m(B^x_t)$ is provable in HA4. So, $\exists x\, m(B)$, i.e. $m'(A)$, is provable in HA4. *q.e.d.*

For the system HA4N and its extensions we can obtain even more economical translations. Let p and p' be the translations from L into $L\Box$ such that $p(A)$ prefixes $\Box$ to every subformula of A save conjunctions, disjunctions and subformulae with and initial existential quantifier, and $p'(A)$ does the same to proper subformulae of A. Then we can prove the following lemma:

LEMMA 3. *In* HA4N *and its extensions we have that*:

$m(A)$ is provable iff $p(A)$ is provable,

$m'(A)$ is provable iff $p'(A)$ is provable.

This is because in HA4N we can prove:

$$\Box(\Box B \wedge \Box C) \leftrightarrow (\Box B \wedge \Box C),$$
$$\Box(\Box B \vee \Box C) \leftrightarrow (\Box B \vee \Box C),$$
$$\Box\exists x\Box B \leftrightarrow \exists x\Box B.$$

As a corollary of Lemmata 1, 2 and 3 we obtain immediately that if A is provable in HA, then $p(A)$ and $p'(A)$ are provable in HA4N.

A further economy in our translations may be achieved for HAS4 and its extensions, since in these systems we can prove $\Box(t_1 = t_2) \leftrightarrow t_1 = t_2$. For the right to left direction of this equivalence we use $t_1 = t_2 \rightarrow (\Box(t_1 = t_1) \rightarrow \Box(t_1 = t_2))$. If we were to take weaker versions of our modal systems where $x = y \rightarrow (A^z_x \rightarrow A^z_y)$ is assumed with the proviso that A is in L (i.e. no $\Box$ occurs in A), then $t_1 = t_2 \rightarrow \Box(t_1 = t_2)$ would not be available, but we could still prove our embeddings of HA below (we prove $\Box(x = y) \rightarrow (m(A^z_x) \rightarrow m(A^z_y))$ in these weaker versions by induction on the complexity of A). To these weaker versions of our modal systems we may also add $\Box(x = y) \rightarrow (A^z_x \rightarrow A^z_y)$ for arbitrary A in $L\Box$, and still have our embeddings (cf. [**14**]).

The advantage of such more economical translations is only relative, because they are less uniform than m and, hence, more difficult to memorize.

Next, we shall use the method of [**9**] in a slightly different version to establish that if $m(A)$ is provable in PAS4, then A is provable in HA.

Let $\neg_C A$ be an abbreviation for $A \rightarrow C$, and $\boxed{C}A$ an abbreviation for $(A \rightarrow C) \rightarrow C$ i.e. for $\neg_C \neg_C A$. Next, let X be a finite nonempty set of formulae of L. Then for $C \in X$, the one-one mapping $(\ )^C_X$ from $L\Box$ into L is defined by the following recursive clauses:

$$
\begin{array}{lll}
(t_1 = t_2)^C_X & \text{is} & \boxed{C}(t_1 = t_2), \\
(A\ \alpha\ B)^C_X & \text{is} & \boxed{C}((A)^C_X\ \alpha\ (B)^C_X), \quad \text{where } \alpha \text{ is } \rightarrow, \wedge \text{ or } \vee, \\
(\neg A)^C_X & \text{is} & \boxed{C}\neg_C (A)^C_X, \\
(\beta A)^C_X & \text{is} & \boxed{C}\beta(A)^C_X, \quad \text{where } \beta \text{ is } \forall x \text{ or } \exists x, \\
(\Box A)^C_X & \text{is} & \boxed{C}\bigwedge\!\!\!\bigwedge_{B \in X}(A)^B_X,
\end{array}
$$

where if $X = \{B_1, \dots , B_n\}$, the abbreviation $\bigwedge\!\!\!\bigwedge_{B \in X}(A)^B_X$ stands for $(A)^{B_1}_X \wedge \dots \wedge (A)^{B_n}_X$.

It is easy to see that $\boxed{C}$ behaves like a necessity operator, in the sense that in HA we have $\Box$1–$\Box$4 with $\Box$ replaced by $\boxed{C}$, as well as the following S5-like principle $\boxed{C}(\boxed{C}A \vee \boxed{C}(\boxed{C}A \rightarrow \boxed{C}B))$ which is like a modal translation of $A \vee (A \rightarrow B)$. Moreover, we have in HA the following theorems:

$$
\begin{array}{l}
\boxed{C}(\boxed{C}A \rightarrow \boxed{C}B) \leftrightarrow (\boxed{C}A \rightarrow \boxed{C}B), \\
\boxed{C}(\boxed{C}A \wedge \boxed{C}B) \leftrightarrow (\boxed{C}A \wedge \boxed{C}B), \\
\boxed{C}\neg_C B \leftrightarrow \neg_C B, \\
\boxed{C}\forall x \boxed{C}B \leftrightarrow \forall x \boxed{C}B,
\end{array}
$$

which show that we could make some economies in the translation $(\ )^C_X$. (Such an economical version of $(\ )^C_X$ is in [**9**], which also differs from our approach by having $\bot$ primitive in L and $L\Box$ instead of $\neg$; see the final part of the second section below).

Then, as in [**9**], by induction on the length of proof of A in PAS4, we can prove the following lemma:

LEMMA 4. *If A is provable in* PAS4, *then for every X and every $C \in X$ we have that $(A)_X^C$ is provable in* HA.

As in [**9**, Lemmata 1.10 and 2.5], by induction on the complexity of A, we can prove the following lemma:

LEMMA 5. *If X contains all the subformulae of A, then in* HA *we can prove* $A \leftrightarrow \bigwedge\!\!\!\bigwedge_{C \in X}(m(A))_X^C$.

This is enough to establish the following lemma, whose analogue we can find in [**9**]:

LEMMA 6. *If $m(A)$ is provable in* PAS4, *then A is provable in* HA.

Proof. Suppose $m(A)$ is provable in PAS4, and let X be the set of subformulae of A. By Lemma 4, for every $C \in X$, in HA we can prove $(m(A))_X^C$. Hence, in HA we have $\bigwedge\!\!\!\bigwedge_{C \in X}(m(A))_X^C$. But then by Lemma 5 in HA we can prove A. *q.e.d.*

We sum up the results of this section in the following theorem:

THEOREM 1. (i) *If* HA4 $\subseteq S \subseteq$ PAS4, *then*

A *is provable in* HA *iff* $m(A)$ *is provable in* S,
iff $m'(A)$ *is provable in* S.

(ii) *If* HA4N $\subseteq$ S $\subseteq$ PAS4, *then*

A *is provable in* HA *iff* $m(A)$ *is provable in* S,
iff $m'(A)$ *is provable in* S,
iff $p(A)$ *is provable in* S,
iff $p'(A)$ *is provable in* S.

The embeddings of this theorem are more interesting when the underlying nonmodal arithmetic of S is classical rather than intuitionistic, because then they indicate how a classical mathematician may translate intuitionism to himself. This is why we should single out the possibility of embedding HA by m and m' into every system between PA4 and PAS4, and also the possibility of embedding HA by m, m', p and p' into every system between PA4N and PAS4.

Theorem 1 leaves open the question whether we can embed HA by our modal translations into a system which, like PAGrz, is stronger than PAS4. This question is interesting because of the following fact, which should be compared with the embedding of HA by p' into systems between PA4N and PAS4, mentioned in Theorem 1 (ii). Let K4N and S4Grz be the modal propositional calculi corresponding to PA4N and PAGrz respectively, i.e. these propositional calculi have the respective modal postulates added to classical propositional logic, omitting $\Box 7$. Next let t be the modal translation, from the language of the Heyting propositional calculus into this language extended with $\Box$, analogous to p'; i.e. t prefixes $\Box$ to every proper subformula save conjunctions and disjunctions. Then it may be shown that the Heyting propositional calculus can be embedded by t in a normal propositional modal logic S iff K4N $\subseteq$ S $\subseteq$ S4Grz (see [**7**]).

2. Modal translations of Peano arithmetic

Consider now the modal axiom-schema:

($\Box 10$) $\Box(\Box A \vee \Box(\Box A \to \Box B))$,

which is a modal translation of $A \vee (A \to B)$. Using this schema we introduce the following systems in $L\Box$:

$$\text{HA5} = \text{HA4} + \Box 10; \qquad \text{HAS5} = \text{HAS4} + \Box 10.$$

In HAS5 we can prove $\Box A \vee \Box(\Box A \to B)$, which is a characteristic S5 principle. (The logics corresponding to HA5 and HAS5 are considered in [**6**] and [**5**] respectively.)

The system PA5 is obtained from HA5 by adding $A \vee (A \to B)$. However, by adding $A \vee (A \to B)$ to HAS5 we do not obtain anything new, as witnessed by the following lemma:

LEMMA 7. *In* HAS5 *we can prove* $\Box A \leftrightarrow A$.

Proof. We proceed by induction the complexity of A.

If A is of the form $t_1 = t_2$, we have $\Box(t_1 = t_2) \leftrightarrow t_1 = t_2$, which is provable in all extensions of HAS4.

If A is of the form $B \to C$, we use $\Box(\Box B \to \Box C) \leftrightarrow (\Box B \to \Box C)$, which from right to left we prove as follows:

$$\dfrac{\dfrac{\dfrac{\Box C \to \Box(\Box B \to \Box C)}{(\Box B \to \Box C) \to (\Box B \to \Box(\Box B \to \Box C))}}{(\Box B \to \Box C) \to ((\Box B \vee \Box(\Box B \to \Box C)) \to \Box(\Box B \to \Box C))}}{(\Box B \to \Box C) \to \Box(\Box B \to \Box C).}$$

If A is of the form $B \wedge C$, $B \vee C$ and $\forall x\, B$, we use the theorems of HA4N mentioned after Lemma 3.

If A is of the form $\neg B$, we use $\Box\neg\Box B \leftrightarrow \neg\Box B$, which from right to left follows immediately from $\Box B \vee \Box(\Box B \to \neg(C \to C))$.

Finally, if A is of the form $\forall x\, B$, we use $\Box\forall x\Box B \leftrightarrow \forall x\Box B$, which from right to left follows from the following instance of the Barcan formula $\forall x\, \Box\Box B \to \Box\forall x\, \Box B$ (a proof of the Barcan formula in HAS5 may be found in [**5**], p. 11). *q.e.d.*

From this lemma it follows immediately that HAS5, PAS5 = HAS5 + $A \vee (A \to B)$ and PAtriv = PA + $\Box A \leftrightarrow A$ are one and the same system. This is a consequence of the facts that all atomic formulae of $L\Box$ are of the form $t_1 = t_2$, that for these formulae we have $\Box(t_1 = t_2) \leftrightarrow t_1 = t_2$, and that only necessity operators in front of atomic formulae are essential in the presence of S5 principles.

With weaker versions of our modal systems where $x = y \rightarrow (A^z_x \rightarrow A^z_y)$ is taken with the proviso that A is in L, and where we may also have $\Box(x = y) \rightarrow (A^z_x \rightarrow A^z_y)$ for arbitrary A in $L\Box$, the systems HAS5 and PAS5 would not collapse into PAtriv, and the embedding of PA with m which we prove below would still obtain.

Next, consider the following system in $L\Box$:

$$\text{HA}\neg\neg = \text{HA} + \Box A \leftrightarrow \neg\neg A.$$

(The logic corresponding to HA¬¬ is investigated model-theoretically in [**11**] and in [**4**], which contains some of the results of [**11**] in a different garb; see also [**6**].) It is not difficult to show that in HA¬¬ we can derive □1–□5 and □10; so, HA5 is included in HA¬¬. Of course, both HA¬¬ and PA5 are included in PAtriv.

That HA5 and HA¬¬ are conservative extensions of HA in L is shown by replacing every □ by ¬¬ in the proof of a formula A of L in one of these modal systems; the result is a proof of A in HA. This entails that HA5 and HA¬¬ are properly included in PA5 and PAtriv respectively. We can also easily show that PA5 and PAtriv are conservative extensions of PA in L. (Note that the version of HA¬¬ with $x = y \rightarrow (A^z_x \rightarrow A^z_y)$ restricted to A in L would have the same theorems as HA¬¬.)

By an easy induction on the length of proof of A we can establish the following lemma:

Lemma 8. *If A is provable in* PA, *then $m(A)$ is provable in* HA5.

The induction needed for this lemma differs from the induction needed for Lemma 1 only in having an additional case which is taken care of by □10.

Since HA5 is an extension of HA4, we could achieve a certain economy in our translation $m(A)$ by omitting □ in front of conjunctions (see the remark after Lemma 1). However, the argument of the proof of Lemma 2 would not go through, and the translation m' is now not available. (Otherwise, since HA5 is contained in HA¬¬, in HA we could prove $\neg\neg A \vee \neg A$.)

As an immediate corollary of Lemma 8 we obtain that if A is provable in PA, then $m(A)$ is provable in HA¬¬. This fact does not differ essentially from the well-known fact that if A is provable in PA, then the formula obtained from A by prefixing ¬¬ to every subformula is provable in HA.

For HA¬¬ and its extensions we can obtain a more economical translation by noting that:

$$\Box(t_1 = t_2) \leftrightarrow t_1 = t_2,$$
$$\Box(\Box B \wedge \Box C) \leftrightarrow (\Box B \wedge \Box C),$$
$$\Box(\Box B \rightarrow \Box C) \leftrightarrow (\Box B \rightarrow \Box C),$$

$$\Box\neg B \leftrightarrow \neg B,$$
$$\Box\forall x\,\Box B \leftrightarrow \forall x\,\Box B$$

are provable in HA¬¬. Of course, for PAtriv the economy can be total: we can omit all necessity operators.

The other direction of our embedding is now immediately available (we have nothing like the complications of the translation $(\ )^C_X$):

LEMMA 9. *If* $m(A)$ *is provable in* PAtriv*, then* A *is provable in* PA.

This follows from the fact that if $m(A)$ is provable in PAtriv, then A is provable in PAtriv, and from the conservativeness of PAtriv with respect to PA.

We sum up the results of this section in the following theorem:

THEOREM 2. *If* HA5 $\subseteq$ S $\subseteq$ PAtriv*, then*

A is provable in PA *iff* $m(A)$ *is provable in* S.

The embeddings of this theorem are more interesting when the underlying nonmodal arithmetic of S is intuitionistic rather than classical, because then they indicate how an intuitionist may translate classical mathematics to himself. This is why we should single out the possibility of embedding PA by m into every system between HA5 and HA¬¬.

The interesting modal embeddings we have considered in this paper are of two types. In embeddings of the first type we have a nonmodal system S′ which can be embedded in a modal system whose nonmodal base is a system S″ which is a proper extension of S′. In embeddings of the second type, S″ can be embedded in a modal extension of S′. Embeddings of HA into modal systems based on PA are of the first type, whereas embedings of PA into modal systems based on HA are of the second type. For both types, one direction of our embeddings, that one which from the provability of A in the nonmodal system infers the provability of the modal translation of A in the modal system, is usually proved by a straightforward induction on the length of proof. The other direction is in principle more difficult to prove for the first type, because for the second type we usually have the following simple procedure. Suppose the modal translation of A is provable in the modal extension based on S′. This modal extension will contain among other modal postulates the modal translations of theorems of S″ missing from S′. Usually, this guarantees that the modal translation of A is provable in S″ plus $\Box A \leftrightarrow A$. Then we infer that A is provable in S″ plus $\Box A \leftrightarrow A$ (using replacement of equivalents), and since this last system is a conservative extension of S″, we have that A is provable in S″. This simple procedure is not available for embeddings of the first type.

Some other systems interesting for intuitionism are also covered by Theorem 2; for example, systems based on intermediate logics. However, not all systems interesting for intuitionism in which we can embed PA by a modal translation are covered by this theorem, as witnessed by the following. Let Johansson's arithmetic JA be the system in L obtained from HA by rejecting $\neg A \to (A \to B)$. It is not difficult to show that if A is provable in PA, then $m(A)$ is provable in the modal system JA5 obtained by extending JA with $\Box 1$–$\Box 5$, $\Box 10$ and $\Box\neg\Box A \to (\Box A \to \Box B)$. Since JA5 is included in HA5, it easily follows that PA may be embedded by m in JA5. This is connected with the fact that PA may be embedded by m in $\mathrm{JA} + \Box A \leftrightarrow \neg\neg A$, which includes JA5 and is included in HA$\neg\neg$ (cf. [**13**]).

Next we shall make some comments on a version of our modal translations induced by taking as primitive in L and $L\Box$ the absurd constant proposition $\bot$ instead of negation. So, let L and $L\Box$ have $\bot$ primitive instead of $\neg$, and as usual let $\neg A$ be defined as $A \to \bot$. Nothing changes essentially in the results presented above if we stipulate that in all our translations, and in particular m, the necessity operator $\Box$ is *not* to be prefixed to the subformula $\bot$. (This is the course followed in [**9**]). However, if for the modal translation m we stipulate that $\Box$ is to be prefixed to every subformula, including $\bot$, then a difference arises, because with $\neg$ primitive and $\bot$ defined as $\neg(B \to B)$ we had that $m(\neg A)$, i.e. $\Box\neg m(A)$, is equivalent to $\Box(m(A) \to \bot)$, whereas with $\bot$ primitive and $\neg$ defined we have $m(\neg A) = \Box(m(A) \to \Box\bot)$.

With this new version of m the schema $\Box 5$, i.e. $\neg\Box\neg(A \to A)$ (which amounts to $\Box\bot \to \bot$), is not needed any more for our embeddings. Namely, we can show that A is provable in HA iff $m(A)$ is provable in HA4 minus $\Box 5$ (as before, from left to right we proceed by induction on the length of proof of A; from right to left we use the fact that HA4 minus $\Box 5$ is contained in HA4, and in HA4 we have $\Box\bot \leftrightarrow \bot$). Something similar happens with the new m and the embedding of PA. It can be shown that A is probable in PA iff $m(A)$ is provable in HA5 minus $\Box 5$. (Note that, as we have indicated in the first section, all the modal postulates of HA5 minus $\Box 5$ are provable in HA when we replace $\Box$ by $\boxed{C}$, whereas $\Box 5$ is not; if $\Box 5$ were provable with this replacement, we would have $((\bot \to C) \to C) \to \bot$ in HA).

The intuitively unsatisfactory feature of this new version of m is that it may "abolish" negation in the translation; namely, the $\neg$ of $\neg A$ is not present anymore in $\Box(m(A) \to \Box\bot)$ in HA4 minus $\Box 5$. On the other hand, negation, as well as all the other logical constants of L, are present in this sense in the old version of m.

To conclude, we may say that a classical mathematician might translate the arithmetic, and presumably other theories, of an intuitionist by introducing a modal operator in his language, and vice versa for the intuitionist translating classical arithmetic. The modal logics in question are not uniquely determined and it would be interesting to characterize exactly the classes of those logics which could be used.

Now, when the intuitionist wants to translate classical mathematics he need not add anything new to his language, since in his basic language L he already has at least one modal operator which could serve for the translation; namely, double negation. Is the same true for the classical mathematician translating intuitionism? Does he already have in his basic language L a modal operator which would do the job?

It is well known that the necessity operator of the propositional logic S4Grz has an interpretation in PA in terms of Gödel's **Bew** predicate, where $\Box A$ means roughly $\mathbf{Bew}(\ulcorner A \urcorner) \wedge A$ (see [**2**], Chapter 13, and [**3**]). It is also known that S4Grz is maximal for PAGrz, in the sense that for every nontheorem A of S4Grz there is an instance A' of A obtained by replacing propositional variables by sentences of L such that A' is not a theorem of PAGrz (see [**10**]). However, even if we could embed HA in PAGrz, this would not yet mean that we have in PA a necessity operator which we could use for embedding HA, and we leave open the question whether there is such an operator.

We conjecture that the inclusion of PA4 in PA4N, of PA4N in PAS4, and of HA5 in HA$\neg\neg$, is proper, but we shall not try to prove this here. That the inclusion of PAS4 in PAGrz is proper follows from the maximality of the propositional logics S4 and S4Grz with respect to these systems (see [**10**]) and from the fact that S4 is properly included in S4Grz.

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