PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série tome 46 (60), 1989, pp. 113–118

THE FORMAL SOLUTION OF MIKUSIŃSKI OPERATIONAL DIFFERENTIAL EQUATIONS WITH VARIABLE COEFFICIENTS

I. E. Sharkawi

Abstract. The formal solution of the homogeneous linear differential equation

$$x^{(n)}(\lambda) + a_{n-1}(\lambda)x^{(n-1)}(\lambda) + \cdots + a_0(\lambda)x(\lambda) = 0, \ \ 0 \le \lambda \le \Lambda \tag{1}$$

is considered. $x(\lambda)$ and $a_i(\lambda)$ are functions which map the interval $[0, \Lambda]$ into the field M of Mikusiński operators [**2**].

1. Introduction

The class C of continuous complex-valued functions of non-negative real variable forms a commutative algebra without zero divisors.

The product is defined as the finite convolution and the sum and scalar product are defined in the usual way.

The quotient field of this algebra is the operator field M od Mikusiński. In this operator field the limit, differentiation and integration are defined (pp. 144 and 183 [**2**]).

Let $f \equiv \{f(t)\}$ denote the representation of $f(t)$ in C, $S = 1/l$ the differential operator, $l = \{l\}$ the integral operator, $I = S^0$ the unit element and let $1/S^\alpha = t^{\alpha-1}/\Gamma(\alpha)$, $(\alpha > 0)$, [**2**]. The theory of operational differential equations has not yet been satisfactorily developed.

Mikusiński proved the uniqueness of the initial value problem for the equation $\sum_{i=0}^{n} a_i x^{(i)}(\lambda) = h(\lambda)$, where a_i are constant operators, and the existence in the special case when $a_i = \sum_{k=0}^{n_i} b_k S^k$, b_k is complex, [**1**], [**3**].

Stanković [**4**] proved the existence and the uniqueness of the equation (1) for $a_i(\lambda) \in C_S$, with the initial conditions

$$x(0) = x'(0) = \cdots = x^{(n-1)}(0) = 1.$$

AMS Subject Classification (1980): Primary 44 A 40

2. Operator spaces $C_S(\lambda)$ and $\overline{C}_S(\lambda)$

Stanković [**5**] has defined and investgated the two spaces $C_S(\lambda)$ and $\overline{C}_S(\lambda)$. We present here only the essentials. $C_S(\lambda)$ is the set of all functions which map the interval $[0,\Lambda]$ into $M\colon \lambda \longrightarrow S^\beta w(\lambda)$, $\beta > 0$, where $w(\lambda) = \{w(\lambda,t)\}$, $w(\lambda,t)$ is continuous complex-valued function defined on $D\colon 0 \le \lambda \le \Lambda$, $0 \le t < \infty$. $\overline{C}_S(\lambda)$ is the set of functions $\lambda \longrightarrow w(\lambda)/F(\lambda)$, where

$$F = \frac{S^{-\beta}}{2\pi i}\int_{\mathrm{Re}\, z=0} z^p \exp(tz - z^\sigma)\, dz, \qquad t \ge 0, \quad 0 < \sigma < 1.$$

3. Preliminary comments and notations

Let us have a differential equation with constant coefficients

$$x^{(n)} + a^0_{n-1}x^{(n-1)}(\lambda) + \cdots + a^0_1 x'(\lambda) + a^0_0 x(\lambda) = 0 \tag{2}$$

for which the corresponding characteristic equation

$$r^n + a^0_{n-1}r^{n-1} + \cdots + a^0_1 r + a^0_0 = 0 \tag{3}$$

has n roots with distinct real parts.

It is interesting to consider cases when $a_k(\lambda) = a^0_k + S^{-\beta} f_k(\lambda)$ where, a^0_k $(k = 0,1,\ldots,n)$ are constants,

$$f_k = \sum_\nu A_{k\nu} e^{i\nu\lambda}, \tag{5}$$

such that for any λ in $[0,\Lambda]$, $|f_k| < N_k < N$ where $A_{k\nu}$ are numerical (real or complex), N_k are positive real numbers and $N = \max\{N_k\}$.

We shall call the finite sums of the type (5) functions of class K.

Let us denote by y and ρ the series:

$$y = 1 + \sum_{n=1}^{\infty} (S)^{-\beta n} y_n \tag{6}$$

$$\rho = r_0 + \sum_{n=1}^{\infty} (S)^{-\beta n} r_n \tag{7}$$

where, y_n $(n = 1,2,\ldots)$ are functions from the set K and r_0 is equal to one of the roots of the equation (3).

Substituting (4) in (1) we get

$$\begin{aligned} &x^{(n)}(\lambda) + a^0_{n-1}x^{n-1}(\lambda) + \cdots + a^0_1 x(\lambda) + a^0_0 x(\lambda) \\ &= -S^{-\beta}\left[f_{n-1}(\lambda)x^{(n-1)}(\lambda) + \cdots + f_1(\lambda)x'(\lambda) + f_0(\lambda)x(\lambda)\right] \end{aligned} \tag{8}$$

which in the operator notation can be written as

$$F\left(\frac{d}{d\lambda}\right)x(\lambda) = -S^{-\beta}L\left(\lambda, \frac{d}{d\lambda}\right)x(\lambda), \tag{9}$$

where $F(d/d\lambda)x$, represents the left-hand-side of the equation (2) and

$$L\left(\lambda, \frac{d}{d\lambda}\right)x = \sum_{k=0}^{n-1} f_k(\lambda)\frac{d^k x}{d\lambda^k}. \tag{10}$$

4. Form of the formal solution of the equation (1)

Theorem. *The formal solution of the homogeneous linear differential equations with coefficients, whose variable parts consist of the functions of class K, have the form*

$$x = ye^{\rho\lambda}, \tag{11}$$

where y and ρ are as in (6) *and* (7).

Proof. Let us show that the function y_k and the numbers ρ_k $(k = 1, 2, \dots)$, in the series (6) and (7) can be obtained in such a way as to satisfy the required condition. By substituting (11) in the equation (9), we obtain

$$F\left(\frac{d}{d\lambda}\right) = -S^{-\beta}L\left(\lambda, \frac{d}{d\lambda}\right)e^{\rho\lambda y}.$$

Rewriting the last equation

$$F\left(\rho + \frac{d}{d\lambda}\right)y = -S^{-\beta}L\left(\lambda, \rho + \frac{d}{d\lambda}\right)y$$

or, representing function y and taking into account the characteristic index ρ, we obtain

$$\begin{aligned} F\left(\sum_{n=0}^{\infty} S^{-n\beta}r_n + \frac{d}{d\lambda}\right)\left(1 + \sum_{n=1}^{\infty} S^{-n\beta}y_n\right) \\ = -S^{-\beta}L\left(\lambda, \sum_{n=0}^{\infty} S^{-n\beta}r_n + \frac{d}{d\lambda}\right)\left(1 + \sum_{n=1}^{\infty} S^{-n\beta}y_n\right). \end{aligned} \tag{12}$$

Let us equate the coefficients of the different powers of $S_{-\beta}$:

1. $(S^{-\beta})^0$: $\quad F\left(r_0 + \frac{d}{d\lambda}\right)1 = F(r_0) = 0,$
2. $(S^{-\beta})^1$: $\quad F\left(r_0 + \frac{d}{d\lambda}\right)y_1 + r_1F'(r_0) = -L\left(\lambda, r_0 + \frac{d}{d\lambda}\right)1 = -L(\lambda, r_0).$

i.e.

$$F\left(r_0 + \frac{d}{d\lambda}\right)y_1 = -L(\lambda, r_0) - r_1F'(r_0). \tag{13}$$

Let us denote the function $L(\lambda, r_0)$ by u_1. Then the equation (13) is reduced to

$$F\left(r_0 + \frac{d}{d\lambda}\right) y_1 = u_1 - r_1 F'(r_0), \tag{14}$$

since $u_1 = \sum_{k=0}^{n-1} f_k(\lambda) r_0^k$ can be expressed as $u_1 = \sum_j \beta_j e^{ij\lambda}$ where $\beta_j = \sum_{k=0}^{n-1} A_{jk} r_0^k$.

We shall satisfy the equation (14) and the required conditions, if we but

$$y_1 = \sum_{j \neq 0} \beta_j \frac{e^{ij\lambda}}{F(r_0 + ij)}, \tag{15}$$

$$r_1 = \frac{\beta_0}{F'(r_0)}, \tag{16}$$

3. Let us equate the coeeficients of $S^{-2\beta}$. Then we have

$$F\left(r_0 + \frac{d}{d\lambda}\right) y_2 = -\frac{1}{2} r_1^2 F''(r_0) - r_1 F'\left(r_0 + \frac{d}{d\lambda}\right) y_1 - r_1 L'_{r_0}(\lambda, r_0)$$
$$-L\left(\lambda, r_0 + \frac{d}{d\lambda}\right) y_1 - r_2 F'(r_0), \tag{18}$$

which contains, apart from known quantities which occur in u_1, other quantities denoted by u_2. The equation (18) can be expressed as

$$F\left(r_0 + \frac{d}{d\lambda}\right) y_2 = u_2 - r_2 F'(r_0). \tag{19}$$

But $u_2 \in K$ so $u_2 = \sum_k C_k e^{ik\lambda}$, the equation (19) and the required conditions will be satisfied, if we substitute

$$y_2 = \sum_{k \neq 0} C_k \frac{e^{ik\lambda}}{F(r_0 + ik)}, \tag{20}$$

$$r_2 = \frac{C_0}{F'(r_0)}. \tag{21}$$

4. Let us show finally that if by the method explained above $y_1, y_2, \ldots, y_n$, $r_1, r_2, \ldots, r_n$ are determined, when all $y_m \in K$ $(m = 1, 2, \ldots, n)$, then we can form y_{n+1} and r_{n+1} where $y_{n+1} \in K$.

Let us examine

$$F\left(\sum_{k=0}^{\infty} S^{-k\beta} r_k + \frac{d}{d\lambda}\right).$$

Here the coefficients $S^{-\beta}, S^{-2\beta} \ldots S^{-n\beta}$ respectively have the form

$$P_1\left(\frac{d}{d\lambda}\right), P_2\left(\frac{d}{d\lambda}\right), \ldots, P_n\left(\frac{d}{d\lambda}\right),$$

where, $P_m(d/d\lambda)$ are some polynomials, $(m = 1, \dots, n)$.

Let us also suppose that in

$$L\left(\lambda, \sum_{n=0}^{\infty} S^{-n\beta} r^n + \frac{d}{d\lambda}\right)$$

the terms from 0 to n have the form

$$\sum_{m=0}^{n} S^{-m\beta} Q_m\left(\lambda, \frac{d}{d\lambda}\right),$$

where $Q_m(\lambda, d/d\lambda)$ are some differential operators of the form

$$\frac{1}{m!}\left\{L_{r_0}^{(k)}\left(\lambda, \sum_{n=0}^{m} S^{-n\beta} r_n + \frac{d}{d\lambda}\right)\right\}.$$

Then the part of the expression

$$S^{-\beta} L\left(\lambda, \sum_{n=0}^{\infty} S^{-n\beta} r_n + \frac{d}{d\lambda}\right) y, \tag{22}$$

which contains terms having power of $S^{-\beta}$ up to $n+1$ will be

$$S^{-\beta}(Q_0 + S^{-\beta} Q_1 + \cdots + S^{-n\beta} Q_n)(1 + S^{-\beta} y_1 + \cdots + S^{-n\beta} y_n),$$

whence we can see that the coefficients of $\overline{S}^{(n+1)\beta}$ in (22) will be written as

$$Q_0 y_n + Q_1 y_{n-1} + \cdots + Q_n \cdot 1. \tag{23}$$

Let us consider the expression

$$y_n(Q_0 - P_1) + y_{n-1}(Q_1 - P_2) + \cdots + y_1(Q_{n-1} - P_n) + 1 Q_n,$$

which contains, besides known quantities u_n, (which are already determined by y_n and r_n), certain terms like y_{n+1}.

Then from (23) we obtain

$$F\left(r_0 + \frac{d}{d\lambda}\right) y_{n+1} = u_{n+1} - r_{n+1} F'(r_0). \tag{24}$$

Since $u_{n+1} \in K$ we have $u_{n+1} = \sum L_p e^{ip\lambda}$, where L_p are constant coefficients.

We shall satisfy the equation (24) and the required condition if we put

$$y_{n+1} = \sum_{p \neq 0} L_p \frac{e^{ip\lambda}}{F(r_0 + ip)}, \quad \text{and} \quad r_{n+1} = \frac{L_0}{F'(r_0)}.$$

Thus we have shown the possibility of the formation of the function y_k and the nubers r_k $(k = 1, 2, \dots)$ by which the formal solution

$$x = \left[\exp\left(\lambda \sum_{k=0}^{\infty} S^{-k\beta} r_k\right)\right] \sum_{k=0}^{\infty} S^{-k\beta} y_k, \quad (y_0 = 1)$$

of the linear differential equation with variable coefficients (4) containing functions of the class K is estabilished.

5. On the convergence of the series (6) and (7)

THEOREM: *If the ordinary power seris of complex variable* Z, $\sum_{k=0}^{\infty} N_k Z^k$, *where* $|y_k| < N_k < N$, *has a positive convergence radius* a, *then the series* $\sum_{k=0}^{\infty} y_k$ *is operationally convergent in every domain* D_0: $0 \le t < t_0$, $0 \le \lambda \le \Lambda$ *for every* $\beta > 0$.

Proof: We have the equalities

$$\frac{1}{s}\left(1 + y_1 \frac{1}{S^\beta} + y_2 \frac{1}{S^{2\beta}} + \cdots\right) = \left\{1 + y_1 \frac{t^\beta}{\Gamma(1+\beta)} + y_2 \frac{t^{2\beta}}{\Gamma(1+2\beta)} + \cdots\right\}.$$

But for every $b > 0$ we have

$$\Gamma(1+n\beta) = \int_0^\infty t^{n\beta} e^{-t}\,dt > \int_b^{1+b} t^{n\beta} e^{-t}\,dt > \int_b^{1+b} b^{n\beta} e^{-1-b}\,dt = b^{n\beta} e^{-1-b}, \quad (n = 0, 1, 2, \ldots).$$

If we take an arbitrary fixed interval $0 \le t \le t_0$, then for $b^\beta = 2t_0^\beta / a$, we shall have in D_0

$$\left|\frac{t^{n\beta}}{\Gamma(1+n\beta)} y_{n+1}\right| < N e^{1+b} \left(\frac{a}{2}\right)^n.$$

Hence it follows that the series (6), as a series of two variables λ and t, is uniformly convergent in every D_0, i.e. it is operationally convergent.

In the same way we can show that, if the series $\sum_{k=0}^{\infty} r_k Z^k$ has a positive convergence radius, then the series $\sum_{k=0}^{\infty} S^{-k\beta} r_k$ as a series of functions of the variable t, is uniformly convergent in $0 \le t \le t_0$, i.e. it is operationally convergent.

REFERENCES

[**1**] S. Drobot et J. Mikusiński, *Sur l'unicité des solutions de l'équations différentielles dans les espaces abstraits*, Studia Math. **11** (1950) 38–40.

[**2**] J. Mikusiński, *Operational calculus*, Pergamon, 1959.

[**3**] J. MIkusiński, *Sur les équations différentielles du calcul opérational et leurs applications aux équations aux dérivées partielles*, Studia Math. **12** (1951) 227–270.

[**4**] B. Stanković, *Operator linear differential equation of order m*, Publ. Inst. Math. (Beograd) **2** (1969) 226–234.

[**5**] B.Stanković, *New results in the theory of Mikusiński operators*, Publ. Inst. Math. (Beograd) **5** (1972), 1–11.

Department of Mathematics
Faculty of Science
Al-Azhar University, Cairo, Egypt

(Received 17 08 1988)