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GEODESIC LINES IN D RECURRENT FINSLER SPACES

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Abstract. A D recurrent Finsler space is defined as a Finsler space in which the absolute differential of the metric tensor is recurrent. For some special cases of the parameter and the vector of recurrency some interesting special cases are obtained. An example is the non-recurrent Finsler space with Cartain connection coefficients. After introducing the so called Y connection [**5**], it is examined in which special case of a D recurrent Finsler space the introduced Y connection will give a recurrent Riemannian space. Finally different kinds of definition of a geodesic line are given. The relation between them and the projective change of the metric function are examined. It is prooved that in a D recurrent Finsler space the geodesic line does not depend on the connection coefficients, but only on the metric function of the space.

1. D recurrent Finsler space. If in a Finsler space $F_n(M, L)$ the metric function $L(x, \dot{x})$ is as usually homogeneous of degree one in $\dot{x}$, and it the metric tensor is defined by

$$g_{\alpha\beta}(x, \dot{x}) = \dot{\partial}_\alpha \dot{\partial}_\beta F(x, \dot{x}), \quad f(x, \dot{x}) = 2^{-1} L^2(x, \dot{x}). \tag{1.1}$$

then the following definition can be given:

Definition 1.1. Finsler F_n is D recurrent if there are vector fields $\lambda^\gamma = \lambda^\gamma(x, \dot{x})$ and $\mu^\gamma = \mu^\gamma(x, \dot{x})$ homogeneous of degree zero in $\dot{x}$, such that

$$Dg_{\alpha\beta} = (\lambda_\gamma dx^\gamma + \mu_\gamma Dl^\gamma) g_{\alpha\beta} = K(x, \dot{x}, dx, Dl) g_{\alpha\beta}, \tag{1.2}$$

where D corresponds to the change of the line element $(x, \dot{x})$ to $(x + dx, \dot{x} + d\dot{x})$ and

$$l^\alpha = L^{-1} \dot{x}^\alpha. \tag{1.3}$$

From

$$Dg_{\alpha\beta} = dg_{\alpha\beta} - (\Gamma^{*\delta}_{\alpha\gamma} + \Gamma^{*\delta}_{\beta\gamma} g_{\alpha\delta}) dx^\gamma - (A^\delta_{\alpha\gamma} g_{\alpha\beta} + A^\delta_{\beta\gamma} g_{\alpha\delta}) Dl^\gamma \tag{1.4}$$

and

$$Dl^\gamma = dl^\gamma + \Gamma^{*\gamma}_{o\beta} dx^\beta + A^\gamma_{o\beta} Dl^\beta, \tag{1.5}$$

AMS Subject Classification (1980): Primary 53 C 60 where "o" means the contradiction by l, we obtain

$$Dg_{\alpha\beta} = g_{\alpha\beta|\gamma}dx^\gamma + g_{\alpha\beta}\mid_\gamma Dl^\gamma \tag{1.6}$$

where

$$g_{\alpha\beta|\gamma} = \partial_\gamma g_{\alpha\beta} - L\dot{\partial}_\delta g_{\alpha\beta}\Gamma^{*\delta}_{o\gamma} - \Gamma^*_{\alpha\beta\gamma} - \Gamma^*_{\beta\alpha\gamma} \tag{1.7}$$

$$g_{\alpha\beta|\gamma} = L\partial_\delta g_{\alpha\beta}(\delta^\delta_\gamma - A^\delta_{o\gamma}) - A_{\alpha\beta\gamma} - A_{\beta\alpha\gamma} \tag{1.8}$$

From the relation

$$g_{\alpha\beta}(x,\dot{x})l^\alpha l^\beta = 1 \tag{1.9}$$

using (1.2) we have

$$\lambda_\gamma dx^\gamma + (\mu_\gamma + 2l_\gamma)Dl^\gamma = 0 \tag{1.10}$$

This is the crucial relation which shows that dx^γ and Dl^γ are not linearly independent. For $\lambda = 0$ and $\mu = 0$, i. e. when $Dg_{\alpha\beta} = 0$, (1.10) reduces to the well-known relation

$$l_\alpha Dl^\alpha = 0 \tag{1.11}$$

in the non-recurrent Finsler space

If we multiply (1.10) with $\theta_{\alpha\beta} = \theta_{\alpha\beta}(x,\dot{x})$ which is homogeneous od degree zero in $\dot{x}, \theta_{\alpha\beta} = \theta_{\beta\alpha}$, and add this to the right-hand side of (1.2) and substitute (1.6) we obtain

$$g_{\alpha\beta|\gamma}dx^\gamma + g_{\alpha\beta}\mid_\gamma Dl^\gamma = \lambda_\gamma(g_{\alpha\beta} + \theta_{\alpha\beta}dx^\gamma + [\mu g_{\alpha\beta} + (\mu_\gamma + 2l_\gamma)\theta_{\alpha\beta}]Dl^\gamma. \tag{1.12}$$

From (1.12) we get

$$g_{\alpha\beta|\gamma} = \lambda_\gamma(g_{\alpha\beta} + \theta_{\alpha\beta}) \tag{1.13}$$

$$g_{\alpha\beta|\gamma} = \mu_\gamma g_{\alpha\beta} + (\mu_\gamma + 2l_\gamma)\theta_{\alpha\beta}. \tag{1.14}$$

In a D recurrent Finsler space we shall suppose that the connection coefficients Γ^* and A are not symmetric, i. e. there exist $(h)h$ and $(v)v$ torsion tensors different from zero. Let us introduce the notation

$$\widetilde{\Gamma}^\beta_{\alpha\gamma} = \Gamma^{*\beta}_{\alpha\gamma} - \Gamma_{\gamma\alpha} \qquad 2\overset{\approx}{\Gamma}_{\alpha\beta\gamma} = \widetilde{\Gamma}_{\beta\gamma\alpha} - \widetilde{\Gamma}_{\beta\gamma\alpha} + \widetilde{\Gamma}_{\gamma\alpha\beta} \tag{1.15}$$

$$\widetilde{A}^\beta_{\alpha\gamma} = A^\beta_{\alpha\gamma} - A^\beta_{\gamma\alpha} \qquad 2\overset{\approx}{A}_{\alpha\beta\gamma} = \widetilde{A}_{\alpha\beta\gamma} - \widetilde{A}_{\beta\gamma\alpha} + \widetilde{A}_{\gamma\alpha\beta} \tag{1.16}$$

From (1.13) and (1.15) we obtain

(1.17)

$$(a) \quad \Gamma^*_{\alpha\beta\gamma} =' \Gamma_{\alpha\beta\gamma} - 2^{-1}[\lambda_\gamma(g_{\alpha\beta} + \theta_{\alpha\beta}) + \lambda_\alpha(g_{\beta\gamma} + \theta_{\beta\gamma}) - \lambda_\beta(g_{\gamma\alpha} + \theta_{\gamma\alpha})] + \overset{\approx}{\Gamma}_{\alpha\beta\gamma},$$

where

$$(b)\qquad {}'\Gamma_{\alpha\beta\gamma} = \gamma_{\alpha\beta\gamma} - 2^{-1}L(\dot\partial_\delta g_{\alpha\beta}\Gamma^{*\delta}_{o\gamma} + \dot\partial_\delta g_{\beta\gamma}\Gamma^{*\delta}_{o\alpha} - \dot\partial_\delta g_{\gamma\alpha}\Gamma^{*\delta}_{o\beta})$$

($\gamma_{\alpha\beta\gamma}$ is the Christoffel symbol)

$$(c)\quad \Gamma^*_{o\beta\gamma} = \gamma_{o\beta\gamma} - 2^{-1}\dot\partial_\delta g_{\beta\gamma}\Gamma^{*\delta}_{oo} - 2^{-1}[\lambda_\gamma(l_\beta + \theta_{o\beta}) + $$
$$+ \lambda_o(g_{\beta\gamma} + \theta_{\beta\gamma}) - \lambda_\beta(l_\gamma + \theta_{\gamma o})] + \overset{\approx}{\Gamma}_{o\beta\gamma}$$

$$(d)\quad \Gamma^*_{o\beta o} = \gamma_{o\beta o} - 2^{-1}[2\lambda_o(l_\beta + \theta_{o\beta}) - \lambda_\beta(1 + \theta_{oo})] + \overset{\approx}{\Gamma}_{o\beta o};$$

(1.18)

$$(a)\quad A_{\alpha\beta\gamma} = {}' A_{\alpha\beta\gamma} - 2^{-1}[\mu_\gamma g_{\alpha\beta} + (\mu_\gamma + 2l_\gamma)\theta_{\alpha\beta} + $$
$$+ \mu_\alpha g_{\beta\gamma} + (\mu_\alpha + 2l_\alpha)\theta_{\beta\gamma} - \mu_\beta g_{\alpha\gamma} - (\mu_\beta + 2l_\beta)\theta_{\alpha\gamma}] + \overset{\approx}{A}_{\alpha\beta\gamma},$$

where

$$(b)\quad {}'A_{\alpha\beta\gamma} = 2^{-1}L[\dot\partial_\delta g_{\alpha\beta}(\delta^\delta_\gamma - A^\delta_{o\gamma}) + \dot\partial_\delta g_{\beta\gamma}(\delta^\delta_\alpha - A^\delta_{o\alpha}) - \dot\partial_\delta g_{\gamma\alpha}(\delta^\delta_\beta - A^\delta_{o\beta})]$$

$$(c)\quad A_{o\beta\gamma} = 2^{-1}L\dot\partial_\delta g_{\beta\gamma}(-A^\delta_{oo}) - 2^{-1}[\mu_\gamma l_\beta + (\mu_\gamma + 2l_\gamma)\theta_{o\beta} + $$
$$+ \mu_o g_{\beta\gamma} + (\mu_o + 2)\theta_{\beta\gamma} - \mu_\beta l_\gamma - (\mu_\beta + 2l_\beta)\theta_{o\gamma}] + \overset{\approx}{A}_{o\beta\gamma}$$

$$(d)\quad A_{o\beta o} = 2^{-1}\{2[\mu_o l_\beta + (\mu_o + 2)\theta_{o\beta}] - [\mu_\beta + (\mu_\beta + 2l_\beta)\theta_{oo}]\} + \overset{\approx}{A}_{o\beta o}.$$

From (1.15) and (1.16) it is clear that

$$\overset{\approx}{\Gamma}_{\alpha\beta\gamma} - \overset{\approx}{\Gamma}_{\gamma\beta\alpha} = \overset{\sim}{\Gamma}_{\alpha\beta\gamma}, \qquad \overset{\approx}{A}_{\alpha\beta\gamma} - \overset{\approx}{A}_{\gamma\beta\alpha} = \tilde{A}_{\alpha\beta\gamma}.$$

For generalized Wagner connection coefficients we have

$$(1.19)\qquad \overset{\sim}{\Gamma}{}^\beta_{\alpha\gamma} = \delta^\beta_\alpha p_\gamma - \delta^\beta_\gamma p_\alpha, \qquad \tilde{A}{}^\beta_{\alpha\gamma} = \delta^\beta_\alpha q_\gamma - \delta^\beta_\gamma q_\alpha,$$

where $p_\gamma = p_\gamma(x,\dot x)$ and $q_\gamma = q_\gamma(x,\dot x)$ are convariant vector fields homogeneous of degree zero in $\dot x$. From (1.19) it follows that

$$(1.20)\qquad \overset{\sim}{\Gamma}\alpha\beta\gamma = g_{\alpha\beta}p_\gamma - g_{\gamma\beta}p_\alpha, \qquad \tilde{A}_{\alpha\beta\gamma} = g_{\alpha\beta}q_\gamma - g_{\beta\gamma}q_\alpha,$$

From (1.15), (1.16) and (1.20) we obtain

$$(1.21)\qquad (a)\quad \overset{\approx}{\Gamma}_{\alpha\beta\gamma} = \overset{\sim}{\Gamma}_{\alpha\gamma\beta} = g_{\gamma\alpha}p_\beta - g_{\beta\gamma}p_\alpha$$

$$(b)\quad \overset{\approx}{\Gamma}_{o\beta\gamma} = l_\gamma p_\beta - p_o g_{\beta\gamma},$$

$$(c)\quad \overset{\approx}{\Gamma}_{o\beta o} = p_\beta - p_o l_\beta;$$

$$(1.22)\qquad (a)\quad \overset{\approx}{A}_{\alpha\beta\gamma} = \tilde{A}_{\alpha\gamma\beta} = g_{\gamma\alpha}q_\beta - g_{\beta\gamma}q_\alpha$$

$$(b)\quad \overset{\approx}{A}_{o\beta\gamma} = l_\gamma q_\beta - q_o g_{\beta\gamma},$$

$$(c)\quad \overset{\approx}{A}_{o\beta o} = q_\beta - q_o l_\beta.$$

Definition 1.2. A D recurrent Finsler space in which the $(h)h$ and $(v)v$ torsion tensors $\widetilde{\Gamma}$ and $\widetilde{A}$ are determined by (1.21) and (1.22) will be called $D(W)$ space.

In this space the connection coefficients are determined by (1.17), (1.18), (1.21) and (1.22).

THEOREM 1.1. *In a D recurent Finsler space the metric tensor is $|$ and $|$ recurrent if*

$$\theta_{\alpha\beta}(x\dot{x}) = k(x,\dot{x})g_{\alpha\beta}(x,\dot{x}), \tag{1.23}$$

where $k(x,\dot{x})$ is a scalar function homogeneous of degree zero in $\dot{x}$. Then we have

$$g_{\alpha\beta|\gamma} = (1+k)\lambda_\gamma g_{\alpha\beta} \tag{1.24}$$

$$g_{\alpha\beta|\gamma} = [(1+k)\mu_\gamma + 2kl_\gamma]g_{\alpha\beta}, \tag{1.25}$$

i. e. the $|$ and $|$ vectors of recurrency are $(1+k)\lambda_\gamma$ and $(1+k)\mu_\gamma + 2kl_\gamma$ respectively.

Proof. It iz evident that (1.2)$\Leftrightarrow$ (1.13)$\wedge$ (1.14). Substituiting (1.23) into (1.13) and (1.14) we obtain (1.24) and (1.25) from which the statement follows.

Definition 1.3. A D recurrent Finsler space in which (1.23) is valid i. e. in which (1.24) and (1.25) hold will be called a $D(k)$ recurrent Finsler space.

THEOREM 1.2. *Every $D(k)$ recurrent Finsler space in D recurrent.*

Proof. Substituing (1.24) and (1.25) into (1.6) and using (1.10) we obtain the statement.

In a $D(k)$ recurrent Finsler space the connectin coefficients have the form

(1.26)

$$(a)\ \Gamma^*_{\alpha\beta\gamma} = {}'\Gamma_{\alpha\beta\gamma} + 2^{-1}(1+k)(\lambda_\gamma g_{\alpha\beta} + \lambda_\alpha g_{\beta\gamma} - \lambda_\beta g_{\gamma\alpha}) + \overset{\approx}{\Gamma}_{\alpha\beta\gamma}$$

$$(b)\ \Gamma^*_{o\beta\gamma} = \gamma_{o\beta\gamma} + 2^{-1}\dot{\partial}_\delta g_{\beta\gamma}\Gamma^{*\delta}_{oo} - 2^{-1}(1+k)(\lambda_\gamma l_\beta + \lambda_o g_{\beta\gamma} - \lambda_\beta l_\gamma) + \overset{\approx}{\Gamma}_{o\beta\gamma}$$

$$(c)\ \Gamma^*_{o\beta o} = \gamma_{o\beta o} - 2^{-1}(1+k)(2\lambda_o l_\beta - \lambda_\beta) + \overset{\approx}{\Gamma}_{o\beta o}$$

(1.27)

$$(a)\ A^*_{\alpha\beta\gamma} = {}'A_{\alpha\beta\gamma} + 2^{-1}(1+k)(\mu_\gamma g_{\alpha\beta} + \mu_\alpha g_{\beta\gamma} - \mu_\beta g_{\gamma\alpha}) - $$
$$-k(l_\gamma g_{\alpha\beta} + l_\alpha g_{\beta\gamma} - l_\beta g_{\alpha\gamma}) + \overset{\approx}{A}_{\alpha\beta\gamma}$$

$$(b)\ A_{o\beta\gamma} = -2^{-1}L\dot{\partial}_\delta g_{\beta\gamma}A^\delta_{oo} - 2^{-1}(1+k)(\mu_\gamma g_{\alpha\beta} + \lambda_\alpha l_\beta + \mu_o g_{\beta\gamma} - \mu_\beta l_\gamma) - kg_{\beta\gamma} + \overset{\approx}{A}_{o\beta\gamma}$$

$$(c)\ A_{o\beta o} = -2^{-1}(1+k)(2\mu_o l_\beta - \mu_\beta) - kl_\beta + \overset{\approx}{A}_{o\beta o}$$

Definition 1.4. A $D(k)$ recurrent Finsler space in which the $(h)h$ and $(v)v$ torsion tensors are given by (1.21) and (1.22) or a $D(W)$ space in which (1.23) holds, will be called a $D(k)(W)$ space.

The connection coefficients in a $D(k)(W)$ space are determined by (1.26), (1.27), (1.21) and (1.22).

THEOREM 1.3. *In $D(k)$ recurrent Finsler space for $k=-1$ we have*

$$(1.28)\qquad (a)\quad g_{\alpha\beta|\gamma}=0 \qquad (b)\quad g_{\alpha\beta|\gamma}=-2l_\gamma g_{\alpha\beta},$$

$$(1.29)\qquad Dg_{\alpha\beta}=-\,2l_\gamma g_{\alpha\beta}Dl^\gamma=(\lambda_\gamma dx^\gamma+\mu_\gamma Dl^\gamma)g_{\alpha\beta}.$$

Proof. (1.28) follows from (1.24), (1.25) and $k=-1$. (1.29) follows from (1.28) and (1.10).

In a $D(k=-1)$ recurrent Finsler space the connection coefficients (1.26) and (1.27) reduce to the form

$$(1.30)\qquad \begin{aligned} &(a)\quad \Gamma^*_{\alpha\beta\gamma}={}'\,\Gamma_{\alpha\beta\gamma}+\overset{\approx}{\Gamma}_{\alpha\beta\gamma}\\ &(b)\quad \Gamma^*_{o\beta\gamma}=\gamma_{o\beta\gamma}-2^{-1}\dot\partial_\delta g_{\beta\gamma}\Gamma^{*\delta}_{oo}+\overset{\approx}{\Gamma}_{o\beta\gamma},\\ &(c)\quad \Gamma^*_{o\beta o}=\gamma_{o\beta o}+\overset{\approx}{\Gamma}_{o\beta o}\end{aligned}$$

$$(1.31)\qquad \begin{aligned} &(a)\quad A_{\alpha\beta\gamma}={}'\,A_{\alpha\beta\gamma}+l_\alpha g_{\beta\gamma}-l_\beta g_{\alpha\gamma})+\overset{\approx}{\Gamma}_{\alpha\beta\gamma},\\ &(b)\quad A_{o\beta\gamma}=-2^{-1}L\dot\partial_\delta g_{\beta\gamma}A^{*\delta}_{oo}+g_{\beta\gamma}+\overset{\approx}{A}_{o\beta\gamma},\\ &(c)\quad A_{o\beta o}=l_\beta+\overset{\approx}{\Gamma}_{o\beta o}.\end{aligned}$$

Definition 1.5. A space $D(k=-1)$ in which the generalized Wagner torsion tensors determined by (1.24) and (1.25) are used or a $D(k)(W)$ space in which $k=-1$ will be called a $D(k=-1)(W)$ space.

The connection coefficients in a $D(k=-1)(W)$ space are determined by (1.30), (1.31), (1.21) and (1.22).

Definition 1.6 A $D(k=-1)$ recurrent Finsler space in which we prescribe the torsion free connection coefficients, i. e. which

$$(1.32)\qquad \overset{\approx}{\Gamma}_{\alpha\beta\gamma}=0,\qquad \overset{\approx}{A}_{\alpha\beta\gamma}=0$$

hold, will be called a $D(k=-1)(TF)$ space

THEOREM 1.4. *In a $D(k=-1)(TF)$ space we have*

$$(1.33)\qquad \Gamma^*_{\alpha\beta\gamma}={}'\,\Gamma_{\alpha\beta\gamma},\quad \Gamma^*_{o\beta\gamma}=\gamma_{o\beta\gamma}-2^{-1}\dot\partial_\delta g_{\beta\gamma}\Gamma^{*\delta}_{oo},\qquad \Gamma^*_{o\beta o}=\gamma_{o\beta o},$$

(1.34)

$$A_{\alpha\beta\gamma}={}'\,A_{\alpha\beta\gamma}+l_\gamma g_{\alpha\beta}+l_\alpha g_{\beta\gamma}-l_\beta g_{\alpha\gamma},\quad A_{o\beta\gamma}=g_{\beta\gamma}\Rightarrow A^\beta_{o\gamma}=\delta^\beta_\gamma,\quad A_{o\beta o}=l_\beta.$$

Proof. The proof follows from (1.30) and (1.31) if in these formulae we substituite from (1.32) $\overset{\approx}{\Gamma}_{\alpha\beta\gamma}=0$ and $\overset{\approx}{A}_{\alpha\beta\gamma}=0$. We can see that $\Gamma^*_{\alpha\beta\gamma}$ is the

Cartain connection coefficient which follows from (1.28 a) and (1.33), but $A_{\alpha\beta\gamma} \neq 2^{-1}L\dot{\partial}_\gamma g_{\alpha\beta}$ because of (1.28 b).

In a $D(k=-1)(TF)$ space Dl^δ cannot be determined by (1.5). In this space (1.34) gives that (1.5) reduces to $dl^\delta + \Gamma^{*\delta}_{o\beta} dx^\beta = 0$ from which $d\dot{x}^\delta$ can be expressed and Dl^δ should satisfy (1.10).

2. Metrical spaces. *Definition* 2.1. A Finsler space will be called D, $|$ and $|$ metrical if $Dg_{\alpha\beta} = 0$, $g_{\alpha\beta|\gamma} = 0$ and $g_{\alpha\beta|\gamma} = 0$ respectively. (See (1.7), (1.8)).

THEOREM 2.1. *If a Finsler space is D metrical, i. e.*

$$Dg_{\alpha\beta} = 0 \tag{2.1}$$

then from (2.1) *follows*

$$g_{\alpha\beta|\gamma} = \lambda_\gamma (g_{\alpha\beta\gamma} + \theta_{\alpha\beta}), \qquad g_{\alpha\beta|\gamma} = \mu_\gamma (g_{\alpha\beta} + \theta_{\alpha\beta}), \tag{2.2}$$

where $\theta_{\alpha\beta} = \theta_{\alpha\beta}(x, \dot{x})$ is homogeneous of degree zero in $\dot{x}$ and $\theta_{\alpha\beta} = \theta_{\alpha\beta}$.

Proof. From $Dg_{\alpha\beta} = 0$ and (1.9) it follows (1.11) i. e. $l_\alpha D!^\alpha = 0$; so (1.10) reduces to $\lambda_\gamma dx^\gamma + \mu_\gamma Dl^\gamma = 0$. From

$$Dg_{\alpha\beta} = g_{\alpha\beta|\gamma} dx^\gamma + g_{\alpha\beta|\gamma} Dl^\gamma = \lambda_\gamma (g_{\alpha\beta} + \theta_{\alpha\beta}) dx^\gamma + \mu_\gamma (g_{\alpha\beta} + \theta_{\alpha\beta} Dl^\gamma$$

(2.2) follows.

THEOREM 2.2. *In a $D(k)$ recurrent Finsler space the metric tensor is D, $|$ and $|$ metrical, i. e.*

$$Dg_{\alpha\beta\gamma} = 0, \qquad g_{\alpha\beta|\gamma} = 0, \qquad g_{\alpha\beta} \,|_\gamma = 0, \tag{2.3}$$

when

$$(\lambda_\gamma = 0) \wedge (\mu_\gamma = -2k(1+k)^{-1} l_\gamma). \tag{2.4}$$

Proof. The proof is obvious from (1.24), (1.25) and (1.6).

THEOREM 2.3. *A $D(k)(TF)$ space in which* (2.4) *holds is a non-recurrent Finsler space supplied with Cartan connection coefficients.*

Proof. The proof follows from

$$\begin{array}{llll} (a) & \mu_\gamma = -2k(1+k)^{-1} l_\gamma & \Rightarrow \mu_o = -2k(1+k)^{-1} & \\ (b) & \lambda_\gamma = 0 & \Rightarrow \lambda_o = 0 & \\ (c) & \theta_{\alpha\beta} = k g_{\alpha\beta} & \Rightarrow \theta_{o\beta} = k l_\beta \Rightarrow \theta_{oo} = k. & \end{array} \tag{2.5}$$

Substituing (2.5) into (1.17) and (1.18) after some calculation we get

(2.6)

$$(a) \qquad \Gamma^*_{o\beta o} = \gamma_{o\beta o},$$

$$(b) \qquad \Gamma^*_{o\beta\gamma} = \gamma_{o\beta\gamma} - 2^{-1}\dot{\partial}_\delta g_{\beta\gamma}\Gamma^{*\delta}_{oo}$$

$$(c) \qquad \Gamma^*_{\alpha\beta\gamma} = \gamma_{\alpha\beta\gamma} - 2^{-1}L(\dot{\partial}_\delta g_{\alpha\beta}\Gamma^{*\delta}_{o\gamma} + \dot{\partial}_\delta g_{\beta\gamma}\Gamma^{*\delta}_{o\alpha} - \dot{\partial}_\delta g_{\gamma\alpha}\Gamma^{*\delta}_{o\beta});$$

(2.7)

$$(a) \qquad A_{o\beta o} = 0,$$

$$(b) \qquad A_{o\beta\gamma} = 0$$

$$(c) \qquad A_{\alpha\beta\gamma} = 2^{-1}L(\dot{\partial}_\gamma g_{\alpha\beta} + \dot{\partial}_\alpha g_{\alpha\gamma} - \dot{\partial}_\beta g_{\gamma\alpha}) = 2^{-1}\dot{\partial}_\gamma g_{\alpha\beta}.$$

From (2.7. a) it follows that in this case (1.5) reduces to the form

$$Dl^\delta = dl^\delta + \Gamma^{*\delta}_{o\beta}du^\beta; \tag{2.8}$$

so the deflection tensor is equal to zero.

THEOREM 2.4. *A $D(k)(TF)$ space reduces to a non-recurrent Finsler space supplied with Cartan connection coefficients if*

$$k = 0, \qquad \lambda_\gamma = 0, \qquad \mu_\gamma = 0. \tag{2.9}$$

In this space (2.6)–(2.8) *are valid.*

Proof. The proof is obvious from (1.26), (1.27) and (2.9).

3. Y **connection in** D **recurrent Finsler spaces**. The name of these connection coefficients is taken from [**5**]. Let the field of the tangent vector (1.3)

$$\dot{x}^\alpha = Y^\alpha(x) \Rightarrow l^\alpha = L^{-1}(x, Y(x))Y^\alpha(x) \tag{3.1}$$

in the domain of manifold M of a D recurrent Finsler space $F_n(M, L)$ be defined. Then from (3.1)

$$d\dot{x}^\alpha = \partial_\beta Y^\alpha dx^\beta \tag{3.2}$$

and (1.5) has the form

$$\begin{aligned}&[\delta^\alpha_\gamma - L^{-1}(x, Y(x))A^\alpha_{\beta\gamma}(x, Y(x))Y^\beta(x)]Dl^\gamma = \\ &= d[L^{-1}(x, Y(x))Y^\alpha(x)] + L^{-1}(x, Y(x))\Gamma^{*\alpha}_{\beta\gamma}(x, Y(x))Y^\beta(x)dx^\gamma.\end{aligned} \tag{3.3}$$

If we denote by I^δ_α the inverse matrix of

$$\mathcal{J}^\alpha_\gamma = \delta^\alpha_\gamma - A^\alpha_{o\gamma} \qquad \det\left[\mathcal{J}^\alpha_\gamma\right] \neq 0 \tag{3.4}$$

from (3.2),(3.3) and (3.4) we obtain

$$Dl^\delta = I^\delta_\iota Y^\iota_{|\gamma}dx^\gamma, \tag{3.5}$$

where

$$Y^{\iota}_{|\gamma} : Y^{\iota}(\partial_{\gamma} L^{-1} + \dot{\partial}_{\theta} L^{-1} \partial_{\gamma} Y^{\theta}) + L^{-1}\partial_{\gamma} Y^{\iota} + L^{-1}\Gamma^{*\iota}_{\theta\gamma} Y^{\theta}. \tag{3.6}$$

In this case we have a Riemannian metric tensor

$$\overline{g}_{\alpha\beta}(x) = g_{\alpha\beta}(x, Y(x)). \tag{3.7}$$

If we denote by $\overline{D}$ the absolute differential which corresponds to the translation from (x) to $(x + dx)$ from (1.4), (3.5), (3.6) and (3.7) we have

$$\overline{D}\overline{g}_{\alpha\beta} = Dg_{\alpha\beta} = dg_{\alpha\beta} - (\Gamma^{*}_{\alpha\beta\gamma} + \Gamma^{*}_{\beta\alpha\gamma})dx^{\gamma} - (A_{\alpha\beta\delta} + A_{\beta\alpha\delta}) I^{\delta}_{\iota} Y^{\iota}_{|\gamma} dx^{\gamma}, \tag{3.8}$$

where $dg_{\alpha\beta} = (\partial_{\gamma} g_{\alpha\beta} + \dot{\partial}_{\delta} g_{\alpha\beta} \partial_{\gamma} Y^{\delta})dx^{\gamma} = d\overline{g}_{\alpha\beta} = \partial_{\gamma}\overline{g}_{\alpha\beta} dx^{\gamma}$. In (3.8)

$$\Gamma^{*}_{\alpha\beta\gamma}(x, \dot{x}) = \Gamma^{*}_{\alpha\beta\gamma}(x, Y(x)), \qquad A_{\alpha\beta\gamma}(x, \dot{x}) = A_{\alpha\beta\gamma}(x, Y(x)),$$

where Γ^{*} and A are given by (1.17) and (1.18). Formula (3.8) determines in a D recurrent Finsler space a Riemannian Y connection coefficient $F_{\alpha\beta\gamma}$, where

$$F_{\alpha\beta\gamma} = \Gamma^{*}_{\alpha\beta\gamma} - A_{\alpha\beta\delta} I^{\delta}_{\iota} Y^{\iota}_{|\gamma}. \tag{3.9}$$

With connection coefficients so defined, from (3.8) we have

$$\overline{D}\overline{g}_{\alpha\beta} = d\overline{g}_{\alpha\beta} - (F_{\alpha\beta\gamma} + F_{\beta\alpha\gamma})dx^{\gamma},$$

where $F_{\alpha\beta\gamma} = F^{\delta}_{\alpha\gamma}\overline{g}_{\delta\beta}$.

Definition 3.1. A D recurrent Finsler space supplied with Y Riemannian connection coefficients (3.9) will be called a $D(Y)$ space.

Theorem 3.1. *If in a $D(Y)$ space we denote by $\top$ the convariant differential with respect to the Y connection F of a Riemannian tensor $\overline{T}^{\alpha}_{\beta}(x) = T^{\alpha}_{\beta}(x, Y(x))$ and with $|$ and $|$ the h and v convariant differentials with respect to Finslerian connection coefficients $\Gamma^{*}(x, Y(x))$ and $A(x, Y(x))$, we have the relation*

$$\overline{T}^{\alpha}_{\beta}(x)\top_{\gamma} dx^{\gamma} = T^{\alpha}_{\beta}(x, Y(x))_{|\gamma} dx^{\gamma} + T^{\alpha}_{\beta}(x, Y(x)) \mid_{\gamma} Dl^{\gamma} \tag{3.10}$$

where

$$\begin{aligned} &(a) \qquad \overline{T}^{\alpha}_{\beta}\top_{\gamma} = \partial_{\gamma}\overline{T}^{\alpha}_{\beta} + F^{\alpha}_{\delta\gamma} T^{\delta}_{\beta} - F^{\delta}_{\beta\gamma}\overline{T}^{\alpha}_{\delta} \\ &(b) \qquad T^{\alpha}_{\beta|\gamma} = \partial_{\gamma} T^{\alpha}_{\beta} \;\; L\dot{\partial}_{\delta} T^{\alpha}_{\beta} \Gamma^{*\delta}_{o\gamma} + \Gamma^{*\alpha}_{\delta\gamma} T^{*\delta}_{\beta} - \Gamma^{*\delta}_{\beta\gamma} T^{\alpha}_{\delta} \\ &(c) \qquad T^{\alpha}_{\beta} \mid_{\gamma} = L\dot{\partial}_{\delta} T^{\alpha}_{\beta}(\delta^{\delta}_{\gamma} - A^{\delta}_{o\gamma}) + A^{\alpha}_{\delta\gamma} T^{\delta}_{\beta} - A^{\delta}_{\beta\gamma} T^{\alpha}_{\delta}. \end{aligned} \tag{3.11}$$

Proof. Substituting (3.11), (3.9) and (3.5) into (3.10) we obtain identity.

Theorem 3.2. *In a $D(Y)$ space we have*

$$\begin{aligned} \overline{D}\overline{g}_{\alpha\beta} = \overline{g}_{\alpha\beta\top\gamma} dx^{\gamma} &= (\lambda_{\gamma} + \mu_{\delta} I^{\delta}_{\iota} Y^{\iota}_{|\gamma}) dx^{\gamma}\overline{g}_{\alpha\beta} + \\ &+ \big[\lambda_{\gamma} + (\mu_{\delta} + 2L^{-1} Y_{\delta}) I^{\delta}_{\iota} Y^{\iota}_{|\gamma}\big] dx^{\gamma}\theta_{\alpha\beta}, \end{aligned} \tag{3.12}$$

where $\overline{g}_{\alpha\beta\top\gamma} = \partial\overline{g}_{\alpha\beta} - F_{\alpha\beta\gamma} - F_{\beta\alpha\gamma}$.

Proof. In a $D(Y)$ space (1.1), (3.10) and (3.5) we have

$$\begin{aligned}\overline{D}\overline{g}_{\alpha\beta} =&\overline{g}_{\alpha\beta}\top_\gamma dx^\gamma = g_{\alpha\beta|\gamma}dx^\gamma + g_{\alpha\beta}\,|_\gamma\; Dl^\gamma = \\ =&(\lambda_\gamma dx^\gamma + \mu_\gamma Dl^\gamma)g_{\alpha\beta} = (\lambda_\gamma + \mu_\delta I^\delta_\iota Y^\iota_{|\gamma})dx^\gamma g_{\alpha\beta}.\end{aligned} \tag{3.13}$$

Relation (1.10) in a $D(Y)$ space has the form

$$\left[\lambda_\gamma + (\mu_\delta + 2L^{-1}Y_\delta)I^\delta_\iota Y^\iota_{|\gamma}\right]dx^\gamma = 0 \tag{3.14}$$

If we multiply (3.14) with $\theta_{\alpha\beta}(\theta_{\alpha\beta} = \theta_{\beta\alpha}(x, \lambda\dot{x}) = \theta_{\alpha\beta}(x, \dot{x}))$ and add the expression so obtained to the right-hand side of (3.13) we get (3.12).

Definition 3.2. A $D(k)$ recurrent Finsler space in which $\dot{x}^\alpha = Y^\alpha(x)$ i.e. a $D(Y)$ space in which $\theta_{\alpha\beta} = kg_{\alpha\beta}$ will be called a $D(k)(Y)$ space.

In $D(k)(Y)$ space the Riemannian connection coefficients F are determined by (3.9) in which Γ^* and A are given by (1.26) and (1.27) respectively.

Theorem 3.3. *In a $D(k)(Y)$ space the Riemannian metric $\overline{g}$ is $\top$ recurrent with respect to the connection coefficients F in this space. The vector of recurrency is*

$$(1+k)[\lambda_\gamma + \mu_\delta I^\delta_\iota Y^\iota_{|\gamma}] + 2kL^{-1}Y_\delta I^\delta_\iota Y^\iota_{|\gamma}.$$

Proof. In (3.12) we substitute $\theta_{\alpha\beta} = k\overline{g}_{\alpha\beta}$ we have

$$\overline{g}_{\alpha\beta\top\gamma} = [(1+k)(\lambda_\gamma + \mu_\delta I^\delta_\iota Y^\iota_{|\gamma}) + 2kL^{-1}Y_\delta I^\delta_\iota Y^\iota_{|\gamma}]\overline{g}_{\alpha\beta} \tag{3.15}$$

which proves the theorem.

Definition 3.3. A $D(k)(Y)$ space in which $L(x, Y(x)) = 1$ will be called a $D(Y_0)$ space.

In a $D(Y_0)$ space

$$L(x, Y(x)) = 1 \;\Rightarrow\; l^\alpha = Y^\alpha(x) \tag{3.16}$$

$$F_{\alpha\beta\gamma} = \Gamma^*_{\alpha\beta\gamma} - A_{\alpha\beta\delta}I^\delta_\iota Y^\iota_{:\gamma}, \tag{3.17}$$

where

$$Y^\iota_{;\gamma} : \partial_\gamma Y^\iota + \Gamma^{*\iota}_{\alpha\beta}Y^\alpha \tag{3.18}$$

and Γ^* and A are determined by (1.17) and (1.18).

Theorem 3.4. *In a $D(Y_0)$ space*

$$\overline{D}\overline{g}_{\alpha\beta} = \overline{g}_{\alpha\beta\top\gamma}dx^\gamma = (\lambda_\gamma + \mu_\delta I^\delta_\iota Y^\iota_{;\gamma})\overline{g}_{\alpha\beta} + [\lambda_\gamma + (\mu_\delta + 2Y_\delta)I^\delta_\iota Y^\iota_{;\gamma}]\theta_{\alpha\beta}. \tag{3.19}$$

Proof. In $D(Y_0)$ from (3.16) it follows that $Y^\iota_{|\gamma} = Y^\iota_{;\gamma}$. From this fact and (3.12) we obtain (3.19) . Theorem 3.4. is the specal case of theorem 3.2., when $L(x, Y(x)) = 1$.

Definition 3.4. A $D(k)$ recurrent Finsler space in which $\dot{x}^\alpha = Y^\alpha(x)$, $L(x, Y(x)) = 1$ will be called a $D(k)(Y_0)$ space.

In a $D(k)(Y_0)$ space the connection coefficients F are given by (3.17) in which $\Gamma^*_{\alpha\beta\gamma}$ and $A_{\alpha\beta\gamma}$ are determined by (1.26) and (1.27) respecitively.

THEOREM 3.5. *In a $D(k)(Y_0)$ space the Riemannian metric $\overline{g}$ is $\top$ recurrent with respect to the connection coefficient F in this space. The vector of recurrency is given by*

$$\overline{g}_{\alpha\beta\top\gamma} = [(1+k)(\lambda_\gamma + \mu_\delta I^\delta_\iota Y^\iota_{;\gamma}) + 2kL^{-1}Y_\delta I^\delta_\iota Y^\iota_{;\gamma}]\overline{g}_{\alpha\beta}. \tag{3.20}$$

Proof. As in $D(k)(Y)$ space the proof follows directly from (3.15).

THEOREM 3.6. *In a $D(k)(Y)$ and in a $D(k)(Y_0)$ space each of the conditions* (3.21 a) *and* (3.21b)

$$\begin{aligned} &(a) \quad \lambda_\gamma = 0 \wedge \mu_\gamma = 0 \wedge k = 0 \\ &(b) \quad \lambda_\gamma = 0 \wedge \mu_\gamma = -2k(1+k)^{-1} l_\gamma \end{aligned} \tag{3.21}$$

leads to $\overline{g}_{\alpha\beta\top\gamma} = 0$, where $\top$ is the convariant differential with respect to the connection coefficients (3.17)*; Γ^* and A satisfy* (1.26) *and* (1.27) *respectively.*

Proof. The proof follows from (3.15) and (3.20) and the condition of Theorem 3.6.

Definition. 3.5. A $D(k)(TF)$ space in which $\dot{x}^\alpha = Y^\alpha(x)$i. e. a $D(k)(Y)$ space in which the torsions are zero ($\overset{\approx}{\Gamma}_{\alpha\beta\gamma} = 0, \overset{\approx}{A}_{\alpha\beta\gamma} = 0$) will b called a $D(k)(TF)(Y)$ space.

A $D(k)(TF)$ space in which $\dot{x}^\alpha = T^\alpha(X)$ and $L(x)$, $Y(x) = 1$, i. e. a $D(k)(Y_0)$ space in which the torsion tensors are zero will be called a $D(k)(TF)(Y_0)$ space.

THEOREM 3.7. *In a $D(k)(TF)(Y)$ and a $D(k)(TF)(Y_0)$ space in which one of the conditions* (3.21 a) *or* (3,21 b) *hold is $\top$ metrical, i. e., $\overline{g}_{\alpha\beta\top\gamma} = 0$ with respect to the connections $F_{\alpha\beta\gamma}$ which are determined by* (3.9), (2.6), (2.7) *and in $D(k)(TF)(Y)$ we have $Y^\iota_{|\gamma}$ determined by* (3.6) *whereas in $D(k)(TF)(Y_0)$ determined by* (3.18).

This theorem is a special case of Theorem 3.6. when the torsion tensors are zero.

Proof. In Thorem 2.3. and 2.4. we proved that under conditions (3.21 a) and (3.21b) in a $D(k)(TF)$ space the condition coefficients Γ^* and A have the form given by (2.6) and (2.7), i. e., they are Cartain connection coefficients.

Note. The Y connection examined in [**5**] is a special case of the connection in a $D(k)(TF)(Y_0)$ space when (3.21 a) is valid.

4. Geodesic lines in D recurrent Finsler spaces. *Definition* 4.1. The geodesic line of a D recurrent Finsler space is the solution of the variation problem

$$\delta \int_{P_1}^{P_2} F(x, \dot{x}) dt = 0.$$

From (1.1) we obtain

$$F(x, \dot{x}) = 2^{-1} g_{\alpha\beta} \dot{x}^\alpha \dot{x}^\beta = 2^{-1} L^2(x, \dot{x}). \tag{4.1}$$

The Euler Lagrange equation for this problem is

(4.2)

$$(a) \quad \frac{d}{dt}(\dot{\partial}_\delta F) - \partial_\delta F = 0 \Leftrightarrow$$

$$(b) \quad (\partial_\theta \dot{\partial}_\delta F) \dot{x}^\theta + (\dot{\partial}_\theta \dot{\partial}_\delta F) \ddot{x}^\theta - \partial_\delta F = 0 \Leftrightarrow$$

$$(c) \quad L\left(\frac{d}{dt} \ln L\right) \dot{\partial}_\delta L + L[(\dot{\partial}_\delta \partial_\theta L) \dot{x}^\theta + (\dot{\partial}_\delta \dot{\partial}_\theta L) \ddot{x}^\theta - \partial_\delta L] = 0 \Leftrightarrow$$

$$(d) \quad E_\delta(L^2) : L\left(\frac{d}{dt} \ln L\right) \dot{\partial}_\delta L + L\left[\frac{d}{dt}(\dot{\partial}_\delta L) - \partial_\delta L\right] = 0.$$

From (4.1) it follows that:

$$\dot{\partial}_\delta F = g_{\delta\beta} \dot{x}^\beta \Rightarrow$$

$$(\partial_\theta \dot{\partial}_\delta F) \dot{x}^\theta = \partial_\theta g_{\delta\beta} \dot{x}^\beta \dot{x}^\theta \tag{4.3}$$

$$(\dot{\partial}_\theta \dot{\partial}_\delta F) \ddot{x}^\theta = g_{\delta\theta} \ddot{x}^\theta \tag{4.4}$$

$$\partial_\delta F = 2^{-1} \partial_\delta g_{\alpha\beta} \dot{x}^\alpha \dot{x}^\beta. \tag{4.5}$$

From (1.7) and (1.13) it follows that

$$\partial_\theta g_{\delta\beta} = L(\dot{\partial}_\chi g_{\delta\beta}) \Gamma^{*\chi}_{o\theta} + \Gamma^*_{\delta\beta\theta} + \Gamma^*_{\beta\delta\theta} + \lambda_\theta (g_{\delta\beta} + \theta_{\alpha\beta}). \tag{4.6}$$

Substituting (4.6) into (4.7) and (4.5) and then (4.3), (4.4), (4.5) into (4.2 b) we obtain that the Euler Lagrange equation in a D recurrent Finsler space has the form

$$E_\delta(L^2) = g_{\delta\theta} \ddot{x}^\theta + L^2[\Gamma^*_{\delta oo} + \Gamma^*_{o\delta o} + \Gamma^*_{oo\delta} + \lambda_o(l_\delta + \theta_{\delta o}) - 2^{-1}\lambda_\delta - 2^{-1}\theta_{oo}\lambda_\delta] = 0 \tag{4.7}$$

From (1.17) we obtain

(4.8)

$$(a) \quad \Gamma^*_{oo\delta} = \gamma_{oo\delta} - 2^{-1}\lambda_\delta - 2^{-1}\lambda_\delta \theta_{oo} + \overset{\approx}{\Gamma}_{oo\delta}$$

$$(b) \quad \Gamma^*_{\delta oo} = \gamma_{\delta oo} - 2^{-1}\lambda_\delta - 2^{-1}\lambda_\delta \theta_{oo} + \overset{\approx}{\Gamma}_{\delta oo}$$

$$(c) \quad \Gamma^*_{o\delta o} = \gamma_{o\delta o} - \lambda_o(l_\delta + \theta_{o\delta}) + 2^{-1}\lambda_\delta \theta_{oo} + 2^{-1}\lambda_\delta + \overset{\approx}{\Gamma}_{o\delta o}.$$

Substituting (4.8) into (4.7) we obtain that the equation of the geodesic line is

$$E_\delta(L^2) = g_{\delta\theta} \ddot{x}^\theta + L^2(\overset{\approx}{\Gamma}_{\delta oo} - \overset{\approx}{\Gamma}_{oo\delta} + \overset{\approx}{\Gamma}_{o\delta o} + \gamma_{o\delta o}) = 0 \tag{4.9}$$

Using (1.15) we have

$$\tilde{\tilde{\Gamma}}_{\delta oo} - \tilde{\tilde{\Gamma}}_{oo\delta} + \tilde{\tilde{\Gamma}}_{o\delta o} = 2^{-1}(3\tilde{\Gamma}^*_{o\delta o} - \tilde{\Gamma}_{oo\delta} - \tilde{\Gamma}_{\delta oo} = 0 \tag{4.10}$$

Now we can state the following:

THEOREM 4.1. *The geodesic line (defined by Definition* 4.1) *in a D recurent Finsler space is given by the solution of the differential equation*

$$E_\delta(L^2) = g_{\delta\theta}(L^2)\ddot{x}^\theta + 2G_\delta(L^2) = 0 \quad (2G_\delta(L^2) = \gamma_{\alpha\delta\beta}(L^2)\dot{x}^\alpha\dot{x}^\beta) \tag{4.11}$$

i. e. the equation of the geodesic line does not depend on connection coefficients of the space, only on the metric function $L(x, \dot{x})$.

Proof. Substituting (4.10) into (4.9) we obtain (4.11).

Equation (4.11) may be written in the form

$$E_\delta(L^2) = g_{\delta\theta}(L^2)\frac{d^2x^\theta}{dt^2} + \gamma_{\alpha\delta\beta}(L^2)\frac{dx^\alpha}{dt}\frac{dx^\beta}{dt} = 0$$

5. Projective change of metric in a D recurrent Finsler space. *Definition* 5.1. If the geodesic line of the Finsler space $F_n(M, L)$ is the geodesic line of the Finsler space $\overline{F}_n(M, \overline{L})$ and the inverse is also true, then the change $L \to \overline{L}$ of the metric function is called projective.

We shall examine the change of the metric function of the form

$$\overline{L}(x, \dot{x}) = L(x, \dot{x}) + \beta(x, \dot{x}), \tag{5.1}$$

where $\beta(x, \dot{x})$ is homogeneous of degree one in $\dot{x}$. From (5.1) we obtain

$$\overline{L}^2 = L^2 + 2L\beta + \beta^2 \tag{5.2}$$

and according to (4.2) we calculate

$$\begin{aligned}
&(a) \quad \dot{\partial}_\delta\overline{L}^2 = \dot{\partial}_\delta L^2 + 2\beta\dot{\partial}_\delta L + 2L\dot{\partial}_\delta\beta + \dot{\partial}_\delta\beta^2 \\
&(b) \quad (\partial_\theta\dot{\partial}_\delta\overline{L}^2)\dot{x}^\theta = (\partial_\theta\dot{\partial}_\delta L^2)\dot{x}^\theta + 2(\partial_\theta\dot{\partial}_\delta L)\dot{x}^\theta + \\
&\qquad 2(\dot{\partial}_\delta L)(\partial_\theta\beta)\dot{x}^\theta + 2(\partial_\theta L)(\dot{\partial}_\delta\beta)\dot{x}^\theta + 2(\partial_\theta\dot{\partial}_\delta\beta)\dot{x}^\theta + (\partial_\theta\dot{\partial}_\delta\beta^2)\dot{x}^\theta \\
&(c) \quad (\dot{\partial}_\theta\dot{\partial}_\delta\overline{L}^2)\ddot{x}^\theta = (\dot{\partial}_\theta\dot{\partial}_\delta L^2)\ddot{x}^\theta + 2(\dot{\partial}_\theta\dot{\partial}_\delta L)\ddot{x}^\theta\beta + 2(\dot{\partial}_\delta L)(\dot{\partial}_\theta\beta)\ddot{x}^\theta + \\
&\qquad 2(\dot{\partial}_\theta L)(\dot{\partial}_\delta\beta)\ddot{x}^\theta + 2L(\partial_\theta\dot{\partial}_\delta\beta)\ddot{x}^\theta + (\dot{\partial}_\theta\dot{\partial}_\delta\beta^2)\ddot{x}^\theta \\
&(d) \quad -\partial_\delta L^2 = -\partial_\delta L^2 - 2\beta\partial_\delta L - 2L\partial_\delta\beta - \partial_\delta\beta^2.
\end{aligned} \tag{5.3}$$

Using the notation

$$E_\delta(G) = 2^{-1}[(\partial_\theta\dot{\partial}_\delta G)\dot{x}^\theta + (\dot{\partial}_\theta\dot{\partial}_\delta G)\ddot{x}^\theta - \partial_\delta G] \tag{5.4}$$

where $G = G(x, \dot{x})$ is any function homogeneous of degree two in $\dot{x}$, if we add (5.3.b), (5.3.c) and (5.3.d) previously multiplied by 2^{-1}, after some calculations we obtain

$$E_\delta(\overline{L}^2) = (1 + \beta L^{-1})E_\delta(L^2) + (1 + L\beta^{-1})E_\delta(\beta^2) + \\ + L\beta(\dot{\partial}_\delta \ln L - \dot{\partial}_\delta \ln \beta)(d \ln \beta/dt - d \ln l/dt).$$

On the other hand, it is known that

$$\gamma_{\alpha\delta\beta}(L^2)\dot{x}^\alpha \dot{x}^\beta = 2^{-1}(\partial_\alpha g_{\delta\beta} + \partial_\beta g_{\alpha\delta} - \partial_\delta g_{\alpha\beta})\dot{x}^\alpha \dot{x}^\beta = \\ = 2^{-1}(\partial_\alpha \dot{\partial}_\delta \dot{\partial}_\beta L^2 + \partial_\beta \dot{\partial}_\alpha \dot{\partial}_\delta L^2 - \partial_\delta \dot{\partial}_\alpha \dot{\partial}_\beta L^2)\dot{x}^\alpha \dot{x}^\beta.$$

So we have

$$\gamma_{\alpha\delta\beta}(L^2)\dot{x}^\alpha \dot{x}^\beta = 2^{-1}[(\partial_\alpha \dot{\partial}_\delta L^2)\dot{x}^\alpha - \partial_\delta L^2]. \tag{5.5}$$

If we introduce the notations

$$\gamma_{\alpha\delta\beta}(G)\dot{x}^\alpha \dot{x}^\beta = 2^{-1}[\partial_\alpha(\dot{\partial}_\delta G)\dot{x}^\alpha - \partial_\delta G] \tag{5.6}$$

$$g_{\theta\delta}(G) = 2^{-1}\dot{\partial}_\theta \dot{\partial}_\delta G \tag{5.7}$$

then from (5.2) we obtain

$$g_{\theta\delta}(\overline{L}^2) = g_{\theta\delta}(L^2) + 2g_{\theta\delta}(L\beta) + g_{\theta\delta}(\beta^2) \tag{5.8}$$

$$\gamma_{\alpha\beta\delta}(\overline{L}^2)\dot{x}^\alpha \dot{x}^\beta = [\gamma_{\alpha\delta\beta}(L^2) + 2\gamma_{\alpha\delta\beta}(L\beta) + \gamma_{\alpha\delta\beta}(\beta^2)]\dot{x}^\alpha \dot{x}^\beta. \tag{5.9}$$

Using (5.6), (5.7) and (5.4) we have

$$E_\delta(G) = g_{\theta\delta}(G)\ddot{x}^\theta + \gamma_{\alpha\delta\beta}(G)\dot{x}^\alpha \dot{x}^\beta. \tag{5.10}$$

THEOREM 5.1. *If the geodesic line in the Finsler space $F_n(M, L)$ is defined by Definition* 4.1. *then the change of the metric function $L \to \overline{L} = L + \beta$ is projective when $E_\delta(\beta^2) = -2E_\delta(L\beta)$.*

Proof. From (5.8) — (5.10) we obtain

$$E_\delta(\overline{L}^2) = E_\delta(L^2) + 2E_\delta(L\beta) + E_\delta(\beta^2) \tag{5.11}$$

from which the theorem follows.

THEOREM 5.2. *By the change of the metric function $L \to \overline{L}, \overline{L} = \sqrt{L^2 + \beta^2}$ a curve is a geodesic line (in the sence of Definition* 4.1*) of the Finsler space $F_n(M, \sqrt{L^2 + \beta^2})$ if it is at the same time the geodesic line in $F_n(M, L)$ and $F_n(M, \beta)$.*

Proof. From (5.6), (5.7) and $\overline{L}^2 = L^2 + \beta^2$ we get

$$\gamma_{\alpha\delta\beta}(\overline{L}^2)\dot{x}^\alpha \dot{x}^\beta = [\gamma_{\alpha\delta\beta}(L^2) + \gamma_{\alpha\delta\beta}(\beta^2)]\dot{x}^\alpha \dot{x}^\beta \tag{5.12}$$

$$g_{\theta\delta}(\overline{L}^2) = g_{\theta\delta}(L^2) + g_{\theta\delta}(\beta^2) \tag{5.13}$$

From (5.12), (5.13) and (5.10) we obtain

$$E_\delta(\overline{L}^2) = E_\delta(L^2) + E_\delta(\beta^2)$$

from which the theorem follows.

It is more usual to define the geodesic line by

Definition 5.2. The geodesic line in a D recurrent Finsler space is the solution of the variation problem

$$\delta \int_{P_1}^{P_2} L(x, \dot{x}) dt = 0$$

The Euler Lagrange equation for this problem is

$$\overline{E}_\delta(L) = d(\dot{\partial}_\delta L)/dt - \partial_\delta L = (\partial_\theta \dot{\partial}_\delta L)\dot{x}^\theta + (\dot{\partial}_\theta \dot{\partial}_\delta L)\ddot{x}^\theta - \partial_\delta L = 0 \tag{5.14}$$

From (5.5) we obtain

$$\gamma_{\alpha\delta\beta}(L^2)\dot{x}^\alpha \dot{x}^\beta = L(\partial_\theta \dot{\partial}_\delta L)\dot{x}^\theta - L\partial_\delta L + (\partial_\theta L)(\dot{\partial}_\delta L)x^\theta. \tag{5.15}$$

From (5.7) we have

$$g_{\theta\delta}(L^2)\ddot{x}^\theta = (\dot{\partial}_\theta L)(\dot{\partial}_\delta L)\ddot{x}^\theta + L(\dot{\partial}_\alpha \dot{\partial}_\beta L)\ddot{x}^\theta. \tag{5.16}$$

If we add (5.15) and (5.16), and further use (5.14), we get

$$g_{\theta\delta}(L^2) + \gamma_{\alpha\delta\beta}(L^2)\dot{x}^\alpha \dot{x}^\beta = L\overline{E}_\delta(L) + L(\dot{\partial}_\delta L) d \ln L/dt. \tag{5.17}$$

Now we can state

THEOREM 5.3. *The geodesic line of Definition* 4.1 *coincides with the geodesic line of Definition* 5.2 *iff*

$$(\dot{\partial}_\delta L) d \ln L/dt = 0 \tag{5.18}$$

Proof. Using (5.10), (5.17) has the form

$$E_\delta(L^2) = L\overline{E}_\delta(L) + L(\dot{\partial}_\delta L) d \ln L/dt \tag{5.19}$$

from which the theorem follows.

When the normal parameter is used, i. e. when $L(x, dx/ds) = 1$, then (5.18) is satisfied and the geodesic lines of Definition 4.1 and 5.2 coincide.

If we introduce the so called Randers change $L \to \overline{L} = L + \beta$ we have the following

THEOREM 5.4. *In a D recurrent Finsler space the Randers change of the metric function is projective iff*

$$\overline{E}_\delta(\beta) = (\partial_\theta \dot{\partial}_\delta \beta)\dot{x}^\theta + (\dot{\partial}_\theta \dot{\partial}_\delta \beta)\ddot{x}^\theta - \partial_\delta \beta = 0 \tag{5.20}$$

where the geodesic line is defined by Definition 5.2.

Proof. From (5.2) and (5.14) we obtain

$$\overline{E}_\delta(\overline{L}) = \overline{E}_\delta(L) + \overline{E}_\delta(\beta) \tag{5.21}$$

from which the theorem follows.

A special case of Theorem 5.4. is the result of Hashiguchi and Ishijyo [**5**] where $\beta = b_\delta(x) \cdot \dot{x}^\delta$. In this case (5.20) reduces to $(\partial_\theta b_\delta - \partial_\delta b_\theta)\dot{x}^\theta = 0$.

THEOREM 5.5. *Under the change $L \to \overline{L} = L + \beta$ of the metric function in a D recurrent Finsler space we have*

$$2E_\delta(L\beta) = \beta\overline{E}_\delta(L) + L\overline{E}_\delta(\beta) + (\dot{\partial}_\delta L)d\beta/dt + (\dot{\partial}_\delta\beta)dL/dt \tag{5.22}$$

Proof. From (5.19), (5.1), (5.21) and (5.11) we get

$$\begin{aligned}
&E_\delta(\overline{L}^2) =^{(5.19)} \overline{L}\overline{E}_\delta(\overline{L}) + (\dot{\partial}_\delta \overline{L})d\overline{L}/dt =^{(5.1)}\\
&\quad (L+\beta)\overline{E}_\delta(L+\beta) + \dot{\partial}_\delta(L+\beta)d(L+\beta)/dt =^{(5.21)}\\
&\quad L\overline{E}_\delta(L) + \beta\overline{E}_\delta(L) + L\overline{E}_\delta(\beta) + \beta\overline{E}_\delta(\beta) +\\
&\quad (\dot{\partial}_\delta L)dl/dt + (\dot{\partial}_\delta L)d\beta/dt + (\dot{\partial}_\delta\beta)dL/dt + (\dot{\partial}_\delta\beta)d\beta/dt =^{(5.9)}\\
&\quad E_\delta(L^2) + E_\delta(\beta^2) + \beta\overline{E}_\delta(L) + L\overline{E}_\delta(\beta) + (\dot{\partial}_\delta L)d\beta/dt + (\dot{\partial}_\delta\beta)dL/dt =^{(5.11)}\\
&\quad E\delta(\overline{L}^2) - 2E_\delta(L\beta) + \beta\overline{E}_\delta(L) + L\overline{E}_\delta(\beta) + (\dot{\partial}_\delta L)d\beta/dt + (\dot{\partial}_\delta\beta)dL/dt
\end{aligned}$$

from which (5.22) follows.

From (5.22) it follows that under the change of the metric function $L \to \overline{L} = L + \beta$ a curve is the geodesic line (by Definition 4.1) of the space $F_n(M, \sqrt{L\beta})$ iff

$$\beta\overline{E}_\delta(L) + L\overline{E}_\delta(\beta) + (\dot{\partial}_\delta L)d\beta/dt + (\dot{\partial}_\delta\beta)dl/dt = 0$$

If $(\dot{\partial}_\delta L)d\beta/dt + (\dot{\partial}_\delta\beta)dL/dt = 0$ and a curve is the geodesic line in the sense of Definition 5.2. of the space $F_n(M.L)$ and $F_n(M\beta)$ (i.e. $\overline{E}_\delta(L) = 0$ and $\overline{E}_\delta(\beta) = 0$) then it is also the geodesic line in the sense of Definition 4.1. in the space $F_n(M, \sqrt{L\beta})$ (i.e. $E\delta(\sqrt{L\beta}) = 0$).

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