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NECESSARY CONDITIONS IN A PROBLEM OF CALCULUS OF VARIATIONS

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Abstract. Problem of the calculus of variations with Bolza functionals is considered. Constraints are of both types: equalities and inequalities. The Lagrange multipler rule type theorem, which gives necessary conditions for weak optimality, is proved. When applied to the simplest problem of the calculus of variations , this theorem gives that every smooth minimizing function must satisfy the well known Euler equation and also the differential equation

$$(d/dt)\,(L_{\dot{x}}\dot{x} - L) = -L_t.$$

It should be emphasized that both differential equations are obtained under the only condition that integrand L is continuously differentiable.

1. Introduction. We shall start this section with the precise formulation of the general Bolza problem of the calculus of variations, which is the object of our present investigation. After that we shall state the theorem giving necessary conditions for weak optimality.

Let V be an open set from $R \times R^n \times R^r$ and let W be an open set from $R \times R^n \times R \times R^n$. Let functions $f(t, x, u) : V \to R^n$, $L(t, x, u) : V \to R^{m+1}$ and $l(t_0, x_0, t_1, x_1) : W \to R^{m+1}$ be continuous.

The set of processes P is the set of quadruples $(x(\cdot)), u(\cdot), t_0, t_1)$ of two functions and two real numbers, satisfying the following conditions:

1. $x(\cdot) : [t_0, t_1] \to R^n$ is a smooth function,
2. $u(\cdot) : [t_0, t_1] \to R^r$ is a continuous function,
3. $(t, x(t), u(t)) \in V$ for all $t \in [t_0, t_1]$,
4. $(t_0, x(t_0), t_1, x(t_1)) \in W$.

We shall deal with Bolza functionals $B_i : P \to R$, $i = 0, 1, \dots, m$, defined by

$$B_i(x(\cdot), u(\cdot), t_0, t_1) = \int_{t_0}^{t_1} L_i(t, x(t), \dot{x}(t))dt + l_i(t_0, x(t_0), t_1, x(t_1))$$

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where L_i and l_i are components of functions L and l.

General Bolza problem that we shall investigate here is the following extremal problem defined on the set P:

$$
\begin{aligned}
&B_0(x(\cdot), u(\cdot), t_0, t_1) \to \inf;\\
&\qquad B_i(x(\cdot), u(\cdot), t_0, t_1) \leq 0, \quad i = 1, \dots, k,\\
&\qquad B_i(x(\cdot), u(\cdot), t_0, t_1) = 0, \quad i = k+1, \dots, m,\\
&\qquad x(t) = f(t, x(t), u(t)), \quad t \in [t_0, t_1].
\end{aligned}
$$

The admissible process $\hat{x}(\cdot), \hat{u}(\cdot), \hat{t}_0, \hat{t}_1)$ is weakly optimal if $\varepsilon > 0$ exists such that the inequality.

$$
B_0(x(\cdot), u(\cdot), t_0, t_1) \geq B_0(\hat{x}(\cdot), \hat{u}(\cdot), \hat{t}_0, \hat{t}_1)
$$

holds for each admissible process $(x(\cdot), u(\cdot), t_0, t_1)$ satisfying the conditions:

$$
\begin{gathered}
\mid t_0 - \hat{t}_0 \mid < \varepsilon, \quad \mid t_1 - \hat{t}_1 \mid < \varepsilon;\\
\|x(t) - \hat{x}(t)\| < \varepsilon, \ \ \|u(t) - \hat{u}(t)\| < \varepsilon
\end{gathered}
$$

for all $t \in [\hat{t}_0, \hat{t}_1] \cap [t_0, t_1]$.

Let us define the functions $H : V \times R^{n*} \times R^{m+1*} \to R$ and $h : W \times R^{m+1^*} \to R$ by

$$
\begin{aligned}
H(t, x, u, p, \lambda) &= pf(t, x, u) - \lambda L(t, x, u)\\
h(t_0, x_0, t_1, x_1, \lambda) &= \lambda l(t_0, x_0, t_1, x_1).
\end{aligned}
$$

The function H is usually called Hamiltonian function.

Theorem. *Let the functions f, L and l be continuously differentiable. If the process $(\hat{x}(\cdot), \hat{u}(\cdot), \hat{t}_0, \hat{t}_1)$ is weakly optimal, then there exists $\hat{\lambda} = (\hat{\lambda}_0, \dots, \hat{\lambda}_m) \in R^{m+1*} \backslash \{0\}$ and a smooth function $\hat{p}(\cdot) : [\hat{t}_0\, \hat{t}_1] \to R^{n*}$, such that the following conditions hold:*

$$
\begin{array}{ll}
(1)\ \ \hat{\lambda}_i \geq 0, \ \ i = 0, 1, \dots, k; & (6)\ \ \hat{p}(\hat{t}_0) = \hat{h}_{x_0}\\
(2)\ \ \hat{\lambda}\hat{B}_i = 0, \ \ i = 1, \dots, k; & (7)\ \ \hat{p}(\hat{t}_1) = -\hat{h}_{x_1};\\
(3)\ \ \hat{H}_u(t) = 0, \quad t \in [\hat{t}_0, \hat{t}_1]; & (8)\ \ \hat{H}(\hat{t}_0) = -\hat{h}_{t_0};\\
(4)\ \ \hat{p}(t) = -\hat{H}_x(t), \quad t \in [\hat{t}_0, \hat{t}_1]; & (9)\ \ \hat{H}(\hat{t}_1) = \hat{h}_{t_1}.\\
(5)\ \ \hat{H}(t) = \hat{H}_t(t), \quad t \in [\hat{t}_0, \hat{t}_1]; &
\end{array}
$$

Remark. Throughout this paper, we use short notation:

$$
\begin{aligned}
&\hat{B}_i = B_i(\hat{x}(\cdot), \hat{u}(\cdot), \hat{t}_0, \hat{t}_1), \quad \hat{H}_u(t) = H_u(t, \hat{x}(t), \hat{u}(t), \hat{p}(t), \hat{\lambda}),\\
&\hat{h}_{x_0} = h_{x_0}(\hat{t}_0, \hat{x}(\hat{t}_0), \hat{t}_1, \hat{x}(\hat{t}_1), \hat{\lambda}),
\end{aligned}
$$

etc., the meaning of which is quite clear from the three given examples.

In the section 4.1. of the monograph [**1**] the same problem as here was studied and a similar theorem, giving necessary conditions for the weak optimality, was proved. In that theorem continuous partial derivatives with respect to x and u of the functions f and L are requered, while we suppose that functions f and L are continuously differentiable. But, as a result, we obtain one necessary condition more—the differential equation 5.

2. Proof of the theorem. The general Mayer problem is defined to be a special case of the general Bolza problem, the case when $L = 0$. On the other hand, each general Bolza problem could be reduced to the general Mayer problem of the following form:

$$\begin{aligned}
&y_0(t_1) - y_0(t_0) + l_0(t_0, x(t_0), t_1, x(t_1)) \to \inf;\\
&\qquad y_i(t_1) - y_i(t_0) + l_i(t_0, x(t_0), t_1, x(t_1)) \le 0, \quad i = 1, \dots, k,\\
&\qquad y_i(t_1) - y_i(t_0) + l_i(t_0, x(t_0), t_1, x(t_1)) = 0, \quad i = k+1, \dots, m,\\
&\qquad \dot{x}(t) = f(t, x(t), u(t)),\\
&\qquad \dot{y}(t) = L(t, x(t), u(t)).
\end{aligned}$$

If the process $(\hat{x}(\cdot), \hat{u}(\cdot), \hat{t}_0, \hat{t}_1)$ is weakly optimal for the general Bolza problem, then the process $(\hat{x}(\cdot), \hat{y}(\cdot), \hat{u}(\cdot), \hat{t}_0, \hat{t}_1)$ where

$$\hat{y}(t) = \int_{\hat{t}_0}^{t} L(s, \hat{x}(s), \hat{u}(s)) ds,$$

is weakly optimal for the corresponding general Mayer problem. Let us prove this fact. If the process $(x(\cdot), y(\cdot), u(\cdot), t_0, t_1)$ is admissible for the corresponding general Mayer problem, then we have

$$\dot{y}_i(t) = L_i(t, x(t), u(t)),$$

$i = 1, 2, \dots, m$ and therefore

$$\begin{aligned}
&y_i(t_1) - y_i(t_0) + l_i(t_0, x(t_0), t_1, x(t_1)) =\\
&= \int_{t_0}^{t_1} L_i(t, x(t), u(t)) dt + l_i(t_0, x(t_0), t_1, x(t_1)) = B_i(x(\cdot), u(\cdot), t_0, t_1),
\end{aligned}$$

$i = 1, 2, \dots, m$. It follows that the process $(x(\cdot), u(\cdot), t_0, t_1)$ is admisible for the initial general Bolza problem. If the process $(x(\cdot), y(\cdot), u(\cdot) t_0, t_1)$ satisfies the conditions

$$\begin{gathered}
| t_0 - \hat{t}_0 | < \varepsilon, \quad | t_1 - \hat{t}_1 | < \varepsilon;\\
\|x(t) - \hat{x}(t)\| < \varepsilon, \quad \|y(t) - \hat{y}(t)\| < \varepsilon, \ \|u(t) - \hat{u}(t)\| < \varepsilon
\end{gathered}$$

for $t \in [\hat{t}_0, \hat{t}_1] \cap [\hat{t}_0, \hat{t}_1]$, then the following relations are valid

$$\begin{aligned}
&y_0(t_1) - y_0(t_0) + l_0(t_0, x(t_0), t_1, x(t_1)) =\\
&= B_0(x(\cdot), u(\cdot), t_0, t_1 \ge B_0(\hat{x}(\cdot), \hat{u}(\cdot), \hat{t}_0, \hat{t}_1) =\\
&= \hat{y}_0(\hat{t}_1) - \hat{y}_0(\hat{t}_0) + l_0(\hat{t}_0, \hat{x}(\hat{t}_0), \hat{t}_1, \hat{x}(\hat{t}_1)).
\end{aligned}$$

Let us suppose that our theorem holds for the general Mayer problem. The Hamiltonian function for corresponding Mayer problem is the following mapping

$$(t,x,y,u,p,q,\lambda) \to pf(t,x,u) + qL(t,x,u),$$

$(t,x,u) \in V$, $y \in R^{m+1}$, $p \in R^{n*}$, $q \in R^{m+1*}$, $\lambda \in R^{m+1*}$. If we apply our theorem to the corresponding Mayer problem and its optimal process $(\hat{x}(\cdot), \hat{y}(\cdot), \hat{u}(\cdot), \hat{t}_0, \hat{t}_1)$ we get that there exist $\hat{\lambda} = (\hat{\lambda}_0, \hat{\lambda}_1, \dots, \hat{\lambda}_m) \in R^{m+1*}\backslash\{0\}$ and smooth functions $\hat{p}(\cdot) : [\hat{t}_0, \hat{t}_1] \to R^{n*}$ and $\hat{q}(\cdot) : [\hat{t}_0, \hat{t}_1] \to R^{m+1*}$, such that the following conditions are valid:

(1′) $\hat{\lambda}_i \geq 0, \quad i = 0,1,\dots,k;$

(2′) $\hat{\lambda}_i[\hat{y}_i(\hat{t}_1) - \hat{y}_i(\hat{t}_0) + l_i(\hat{t}_0, \hat{t}_1, \hat{x}(\hat{t}_1))] = 0 \;\; i = 1,\dots,k;$

(3′) $\hat{p}(t)\hat{f}_u(t) + \hat{q}(t)\hat{L}_u(t) = 0, \;\; t \in [\hat{t}_0, \hat{t}_1];$

(4′) $\hat{p}(t) = -\hat{p}(t)\hat{f}_x(t) - \hat{q}(t)\hat{L}_x(t), \quad t \in [\hat{t}_0, \hat{t}_1];$

(4″) $\hat{q}(t) = 0, \quad t \in [\hat{t}_0, \hat{t}_1];$

(5′) $\frac{d}{dt}[\hat{p}(t)\hat{f}(t) + \hat{q}(t)\hat{L}(t)] = \hat{p}(t)\hat{f}_t(t) + \hat{q}(t)\hat{L}_t(t), \quad t \in [\hat{t}_0, \hat{t}_1];$

(6′) $\hat{p}(\hat{t}_0) = \hat{\lambda}\hat{l}_{x_0};$ (6″) $\hat{q}(\hat{t}_0) = -\hat{\lambda};$

(7′) $\hat{p}(\hat{t}_1) = -\hat{\lambda}\hat{l}_{x_1};$ (7″) $\hat{q}(\hat{t}_1) = -\hat{\lambda};$

(8′) $\hat{p}(t_0)\hat{f}(t_0) + \hat{q}(\hat{t}_0)\hat{L}(\hat{t}_0) = -\hat{\lambda}\hat{l}_{t_0};$

(9′) $\hat{p}(\hat{t}_1)\hat{f}(\hat{t}_1) + \hat{q}(\hat{t}_1)\hat{L}(\hat{t}_1) = \hat{\lambda}\hat{l}_{t_1}.$

The conditions 1’, 6’ i 7’ coincide with the conditions 1, 6 i 7 of our theorem. Bearing in mind that

$$\hat{y}_i(\hat{t}_1) - \hat{y}_i(\hat{t}_0) = \int_{\hat{t}_0}^{\hat{t}_1} L_i(t, \hat{x}(t), \hat{u}(t))dt,$$

$i = 0,1,\dots,m$, we obtain the condition 2 of our theorem from the condition 2’. From 4”, 6” i 7” it follows that $\hat{q}(t) = -\hat{\lambda}, \; t \in [\hat{t}_0, \hat{t}_1]$. Using this, from 3’, 4’, 5’, 8’ i 9’ we get the conditions 3, 4, 5, 8 i 9 of our theorem. Consequently, it sufficies to prove that our theorem is valid in the case when $L = 0$.

Let us denote by I the interval $[\hat{t}_0, \hat{t}_1]$. The set

$$\Delta = \{(z(\cdot), x(\cdot), u(\cdot)) \in C_1(I) \times C_1^n(I) \times C^r(I) \mid \\ \forall t \in I)(z(t), x(t), u(t)) \in V, \, (z(\hat{t}_0), \, x(\hat{t}_0), \, z(\hat{t}_1), \, x(\hat{t}_1)) \in W\}$$

is an open subset of the Banach space $C_1(I) \times C_1^n(I) \times C^r(I)$ (where $C_q^p(I)$ is the space of functions mapping I into R^p, having continuous derivative of order q). The functionals $\varphi_i : \Delta \to R, \; i = 0,1,\dots,m$, defined by

$$\varphi_i(\xi) = l_i(z(\hat{t}_0), \, x(\hat{t}_0), \, z(\hat{t}_1), \, x, (\hat{t}_1)),$$

where $\xi = (z(\cdot), \, x(\cdot), \, u(\cdot)) \in \Delta$, are continuously differentiable. The operator $\Phi : \Delta \to C^n(I)$, defined by

$$\Phi(\xi)(t) = \dot{x}(t)\dot{z}(t)f(z(t), \, x(t), \, u(t))$$

is continously differentiable too. Let $\hat{\xi} = \hat{z}(\cdot), \hat{x}(\cdot), \hat{u}(\cdot)) \in \Delta$, where the function $\hat{z}(\cdot) : I \to R$ is defined by $\hat{z}(t) = t$. As

$$\Phi_{x(\cdot)}(\hat{\xi})x(\cdot)(t) = \dot{x}(t)\hat{f}_x(t)x(\hat{t}),$$

then, according to the theorem of existence of solution of the linear differential equation, we have $\operatorname{Im} \hat{\Phi}_{x(\cdot)}(\hat{\xi}) = C^n(I)$, and therefore $\operatorname{Im} \Phi'(\hat{\xi}) = C^n(I)$.

Let us prove that $\hat{\xi}$ is a local solution of the problem

$$\begin{aligned} \varphi_0(\xi) \to \inf; \quad \varphi_i(\xi) &\leq 0, \qquad i = 1, \dots, k, \\ \varphi_(\xi) &= 0, \qquad i = k+1, \dots, m, \\ \Phi(\xi) &= 0. \end{aligned}$$

There exist real δ, $0 < \delta < 1$, such that

$$\delta + \omega(\hat{x}(\cdot),\, \delta) < \varepsilon, \qquad \delta + \omega(\hat{u}(\cdot),\, \delta) < \varepsilon.$$

Let $\xi = (z(\cdot),\, x(\cdot),\, u(\cdot))$, be an admissible point from Δ satisfying conditions

$$\|z(\cdot) - \hat{z}(\cdot)\|,\ \|x(\cdot) - \hat{x}(\cdot)\|,\ \|u(\cdot - \hat{u}(\cdot)\| < \delta.$$

From $\|z(\cdot) - \hat{z}(\cdot)\| < 1$ i follows that $\dot{z}(t) > 0$ for all $t \in I$. Therefore, the continuously differentiable inverse $z^{-1}(\cdot) : [z(\hat{t}_0,\, z(\hat{t}_1)] \to I$ exists. Let us consider the the process $(x \circ z^{-1}(\cdot),\, u \circ z^{-1}(\cdot),\, z(\hat{t}_0),\, z(\hat{t}_1))$. Since

$$B_i(x \circ z^{-1}(\cdot),\, u \circ z^{-1}(\cdot),\, z(\hat{t}_0),\, z(\hat{t}_1)) = \varphi_i(\xi),$$

for $i = 0, 1, \dots m$ and

$$\begin{aligned} \frac{d}{dt} x \circ z^{-1}(t) =& \dot{x}(z^{-1}(t)) \frac{d}{dt} z^{-1}(t) \\ =& \dot{z}(z^{-1}(t)) f(z(z^{-1}(t)),\, x(z^{-1}(t)),\, u(z^{-1}(t))) \frac{1}{\dot{z}(z^{-1}(t))} \\ =& f(t, x \circ z^{-1}(t),\, u \circ z^{-1}(t)), \end{aligned}$$

then that process is admissible. As

$$\begin{aligned} \mid z(\hat{t}_0) - \hat{t}_0 \mid = \mid z(t_0) - \hat{z}(\hat{t}_0) \mid < \delta < \varepsilon, \\ \mid z(\hat{t}_1) - \hat{t}_1 \mid = \mid z(t_1) - \hat{z}(\hat{t}_1) \mid < \delta < \varepsilon, \end{aligned}$$

and, for $t \in [\hat{t}_0, \hat{t}_1] \cap [z(\hat{t}_0),\, z(\hat{t}_1)]$, we have

$$\begin{aligned} \|x \circ z^{-1}(t) - \hat{x}(t)\| \leq& \|x(z^{-1}(t)) - \hat{x}(z^{-1}(t))\| + \|\hat{x}(z^{-1}(t)) - \hat{x}(t)\| \\ \leq& \delta + \omega(\hat{x}(\cdot),\, \delta) < \varepsilon, \\ \|u \circ z^{-1}(t) - \hat{u}(t)\| \leq& \|u(z^{-1}(t)) - \hat{u}(z^{-1}(t))\| + \|\hat{u}(z^{-1}(t)) - \hat{u}(t))\| \\ \leq& \delta + \omega(\hat{u}(\cdot),\, \delta) < \varepsilon, \end{aligned}$$

then

$$\varphi_0(\xi) = B_0(x \circ z^{-1}(\cdot),\, u \circ z^{-1}(\cdot),\, z(t_0), z(t_1)) \geq \hat{B}_0 = \varphi_0(\hat{\xi}).$$

All conditions of Lagrange principle for the smooth problem (see, for example, theorem 3.2.1. from [**1**]) are fulfilled. So, we can assert that Lagrange multipliers $\hat{\lambda} \in R^{m+1*}$ and $\hat{y}^* \in C^n(I)^*$, $(\hat{\lambda}, \hat{y}^*) \neq 0$, exist, such that we have

$$\hat{\lambda}_i \geq 0, \ \ i = 0, 1, \dots, k, \ \ \hat{\lambda}_i \hat{l}_i = 0, \ \ i = 1, \dots, k,$$

(conditions 1 and 2), and such that $\hat{\xi}$ is a stationary point of the Lagrange function

$$\hat{\lambda}\varphi + \hat{y}^*\Phi.$$

When we differentiate the Lagrange function with respect to $x(\cdot)$ at the point $\hat{\xi}$, we obtain that

$$\hat{\lambda}\hat{l}_{x_0}(\hat{t}_0) + \hat{\lambda}\hat{l}_{x_1}x(\hat{t}_1) + \hat{y}^*(\dot{x}(t) - \hat{f}_x(t)x(t)) = 0$$

for all $x(\cdot) \in C_1^n(I)$. Let the smooth function $\hat{p}(\cdot) : [\hat{t}_0, \hat{t}_1] \to R^{n*}$ be a solution of the problem

$$\dot{p}(t) = -p(t)\hat{f}_x(t), \quad p(\hat{t}_0) = \hat{\lambda}\hat{l}_{x_0}$$

(conditions and 6). Let $y(\cdot) \in C^n(I)$ and $x \in R^n$. There exists an $(\cdot) \in C_1^n(I)$, such that

$$\dot{x}(t) = \hat{f}_x(t)x(t) + y(t), \quad x(\hat{t}_1) = x.$$

Since

$$\frac{d}{dt}\hat{p}(t)x(t) = -\hat{p}(t)\hat{f}_x(t)x(t) + \hat{p}(t)(\hat{f}_x(t)x(t) + y(t)) = \hat{p}(t)y(t),$$

then

$$\int_{\hat{t}_0}^{\hat{t}_1} \hat{p}(t)y(t)dt = \hat{p}(t)x(t)\Big|_{\hat{t}_0}^{\hat{t}_1} = \hat{p}(\hat{t}_1)x - \hat{\lambda}\hat{l}_{x_0}x(\hat{t}_0).$$

It follows that

$$(\hat{p}(\hat{t}_1) + \hat{\lambda}\hat{l}_{x_1})x + \hat{y}^*y(\cdot) - \int_{\hat{t}_0}^{\hat{t}_1} \hat{p}(t)y(t)dt = 0.$$

Since the preceding equality is valid for all $x \in R^n$ and every $y(\cdot) \in C^n(I)$, then

$$\hat{y}^*y(\cdot) = \int_{\hat{t}_0}^{\hat{t}_1} \hat{p}(t)y(t)dt, \quad \hat{p}(\hat{t}_1) = \hat{\lambda}\hat{l}_{x_1}$$

(condition 7).

When we differentiate the Lagrange function with respect to $u(\cdot)$ at the point $\hat{\xi}$, we get that

$$\hat{y}^*(-\hat{f}_u(t)u(t)) = 0$$

for all $u(\cdot) \in C^r(I)$, i. e. that

$$\int_{\hat{t}_0}^{\hat{t}_1} \hat{p}(t)\hat{f}_u(t)u(t)dt = 0$$

for all $u(\cdot) \in C^r(I)$. It follows that

$$\hat{p}(t)\hat{f}_u(t) = 0$$

for all $t \in [\hat{t}_0, \hat{t}_1]$ (condition 3).

Differentiating the Lagrange function with respect to $z(\cdot)$ at the point $\hat{\xi}$, we obtain that

$$\hat{\lambda}\hat{l}_{t_0} z(\hat{t}_0) + \hat{\lambda}\ l_{t_1} z(\hat{t}_1) - \hat{y}^*(\dot{z}(t)\hat{f}(t) + z(t)\hat{f}_t(t)) = 0$$

for all $z(\cdot) \in C_1(I)$, i. e. that

$$\hat{\lambda}\hat{l}_{t_0} z(\hat{t}_0) + \hat{\lambda}\ l_{t_1} z(\hat{t}_1) \int_{\hat{t}_0}^{\hat{t}_1} \hat{p}(t)(\dot{z}(t)\hat{f}(t) + z(\hat{t})f_t(t))dt = 0$$

for all $z(\cdot) \in C_1(I)$. Since

$$z(\hat{t}_0) = z(\hat{t}_1) - \int_{\hat{t}_0}^{\hat{t}_1} \dot{z}(t)dt,$$

$$\int_{\hat{t}_0}^{\hat{t}_1} z(t)\hat{p}(t)\hat{f}_t(t)dt =$$

$$= z(\hat{t}_1)\int_{\hat{t}_0}^{\hat{t}_1} \hat{p}(t)\hat{f}_t(t)dt - \int_{\hat{t}_0}^{\hat{t}_1} \dot{z}(t)\left(\int_{\hat{t}_0}^{t} \hat{p}(s)\hat{f}_t(s)ds\right)dt,$$

then

$$\left(\hat{\lambda}\hat{l}_{t_0} + \hat{\lambda}\hat{l}_{t_1} - \int_{\hat{t}_0}^{\hat{t}_1} \hat{p}(t)\hat{f}_t(t)dt\right)z(\hat{t}_1) -$$

$$- \int_{\hat{t}_0}^{\hat{t}_1} \dot{z}(t)\left(\hat{p}(t)\hat{f}(t) + \hat{\lambda}\hat{l}_{t_0} - \int_{\hat{t}_0}^{\hat{t}_1} \hat{p}(s)\hat{f}_t(s)ds\right)dt = 0$$

for all $z(\cdot) \in C_1(I)$. It follows that

$$\hat{\lambda}\hat{l}_{t_0} + \hat{\lambda}\hat{l}_{t_1} - \int_{\hat{t}_0}^{\hat{t}_1} \hat{p}(t)\hat{f}_t(t)dt = 0$$

$$\hat{p}(t)\hat{f}(t) + \hat{\lambda}\hat{l}_{t_0} - \int_{\hat{t}_0}^{t} \hat{p}(s)\hat{f}_t(s)ds = 0$$

for all $t \in [\hat{t}_0, \hat{t}_1]$. From the last equation we obtain that

$$\frac{d}{dt}\hat{p}(t)\hat{f}(t) = \hat{p}(t)\hat{f}_t(t)$$

for all $t \in [\hat{t}_0, \hat{t}_1]$ (condition 5) and

$$\hat{p}(\hat{t}_0)\hat{f}(\hat{t}_0) = -\hat{\lambda}\hat{l}_{t_0}$$

(condition 8). And finally

$$\hat{p}(\hat{t}_1)\hat{f}(\hat{t}_1) = -\hat{\lambda}\hat{l}_{t_0} + \int_{\hat{t}_0}^{\hat{t}_1} \hat{p}(t)\hat{f}(t)dt = \hat{\lambda}\hat{l}_{t_1}$$

(condition 9).

As $\hat{\lambda} = 0$ implies $\hat{p}(\cdot) = 0$ which, in its turn implies $\hat{y}^* = 0$, we have that $\hat{\lambda} \neq 0$. $\Box$.

3. Application to the simplest problem of the calculus of variations. We chall consider here the simplest problem of the calculus of variations. First we shall give its precise formulation, we shall show that it is a special case of the general Bolza problem and we shall show that necessary conditions for weak minimum (from the classical calculus of variations) are consequences of the preceding theorem.

Let V be an open set in $R \times R \times R$, let $L(t, x, \dot{x}) : V \to R$ be a continuous function and let $\hat{t}_0$, $\hat{t}_1$, $\hat{x}_0$ and $\hat{x}_1$ be real numbers, $\hat{t}_0 < \hat{t}_1$. The simplest problem of the calculus of variations is the following extremal problem:

$$J(x(\cdot) = \int_{\hat{t}_0}^{\hat{t}_1} L(t, x(t),\ \dot{x}(t))dt \to \inf;\ \ x(\hat{t}_0) = \hat{x}_0,\ x(\hat{t}_1) = \hat{x}_1.$$

We shall investigate that problem of the following set of smooth functions:

$$\{x(\cdot) \in C_1[\hat{t}_0, \hat{t}_1] \mid (\forall t \in [\hat{t}_0, \hat{t}_1])(t, x(t), \dot{x}(t)) \in V\}.$$

The admissible function $\hat{x}(\cdot) \in C_1[\hat{t}_0, \hat{t}_1]$ furnishes the weak minimum if there exists $\varepsilon > 0$ such hat the inequality

$$J(x(\cdot)) \geq J(\hat{x}(\cdot))$$

holds for each admissible function $x(\cdot) \in C_1[\hat{t}_0, \hat{t}_1]$ satisfying inequalities

$$\mid x(t) - \hat{x}(t) \mid < \varepsilon, \qquad \mid \dot{x}(t) - \dot{\hat{x}}(t) \mid < \varepsilon$$

for all $t \in [\hat{t}_0, \hat{t}_1]$.

We can treat the simplest problem as the general Bolza problem if we transform it in the following from

$$\begin{aligned} &\int_{\hat{t}_0}^{\hat{t}_1} L(t, x(t), u(t))dt \to \inf : \\ &\qquad\qquad x(t_0) = \hat{x}_0,\ x(t_1) = \hat{x}_1,\ t_0 = \hat{t}_0,\ t_1 = \hat{t}_1, \\ &\qquad\qquad \dot{x}(t) = u(t). \end{aligned}$$

It is easy to prove that the process $(\hat{x}(\cdot), \dot{\hat{x}}(\cdot),\ \hat{t}_0, \hat{t}_1)$ is weakly optmal for the preceding Bolza problem if the function $\hat{x}(\cdot)$ furnishes the weak minimum for the initial simplest problem.

Let L be continuously differentiable function. Let $\hat{x}(\cdot)$ be the weak solution of the simplest problem. According to the preseding theorem, there exist $\hat{\lambda} \in R^{5*}\setminus\{0\}$ and smooth function $\hat{p}(\cdot) : [\hat{t}_0, \hat{t}_1] \to R$, such that

(3) $\hat{H}_u(t) = 0$;

(4) $\dot{\hat{p}}(t) = -\hat{H}_x(t)$;

(5) $\dot{\hat{H}}(t) = \hat{H}_t(t)$;

(6) $\hat{p}(\hat{t}_0) = \hat{\lambda}_1$;

(7) $\hat{p}(\hat{t}_1) = -\hat{\lambda}_2$;

(8) $\hat{H}(\hat{t}_0) = -\hat{\lambda}_3$;

(9) $\hat{H}(\hat{t}_1) = \hat{\lambda}_4$

hold, where

$$H(t,\, x,\, u,\, p,\, \lambda) = pu - \lambda_0 L(t,\, x,\, u).$$

From 3 we get that

$$\hat{p}(t) = \hat{\lambda}_0 \hat{L}_{\dot{x}}(t).$$

If we had $\hat{\lambda}_0 = 0$, there would be $\hat{p}(\cdot) = 0$ and $\hat{H}(\cdot) = 0$, and then from 6, 7, 8 and 9 we would have $\hat{\lambda}_1 = \hat{\lambda}_2 = \hat{\lambda}_3 = \hat{\lambda}_4 = 0$, which is impossible. Therefore, we can suppose that $\hat{\lambda}_0 = 1$. Hence

$$\hat{p}(t) = \hat{L}_{\dot{x}}(t).$$

From the preceding equation and from 4 and 5 we get the following two equations, well known from the classical calculus of variations

$$\frac{d}{dt}\hat{L}_{\dot{x}}(t) = \hat{L}_x(t)$$
$$\frac{d}{dt}[\hat{L}_{\dot{x}}(t)\dot{\hat{x}}(t) - \hat{L}(t)] = -\hat{L}_t(t).$$

Now we can summarize our reasoning in the following

THEOREM. *Let the function L be continuously differentiale. If the function $\hat{x}(\cdot) \in C_1[\hat{t}_0, \hat{t}_1]$ furnishes the weak minimum in the simplest problem of the calculus of variations, then the following equations*

$$\frac{d}{dt}\hat{L}_{\dot{x}}(t) = \hat{L}_x(t)$$
$$\frac{d}{dt}[\hat{L}_{\dot{x}}(t)\dot{\hat{x}}(t) - \hat{L}(t)] = -\hat{L}_t(t).$$

hold on $[\hat{t}_0, \hat{t}_1]$.

REFERENCES

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