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THE LINEAR OPTIMAL CONTROL PROBLEM WITH VARIABLE ENDPOINTS

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Abstract. A maximum principle, a uniqueness theorem and an existence theorem for the linear optimal control problem with variable endpoints and with general class of admissible controls are proved.

1. Introduction

Let us begin with framework of the optimal control problem which we are going to study in this paper. The control set U is an arbitrary convex compact set in R^r. Control is a function which maps some closed interval of the real line into the control set. We shall deal with the class D of so-called admissible controls. This is the class of controls which satisfies the following conditions:

1. each admissible control is measurable,

2. each piecewise constant control is admissible,

3. if the control $u(\cdot) : [t_0, t_1] \to U$ is admissible, then the control $u'(\cdot) : [t_0 + h, t_1 + h] \to U$ defined by $u'(t) = u(t - h)$ is admissible too,

4. if the control $u(\cdot) : [t_0, t_1] \to U$ is admissible, then its restriction on an arbitrary closed subinterval of $[t_0, t_1]$ is admissible too,

5. if the restrictions of the control $u(\cdot) : [t_0, t_1] \to U$ on $[t_0, \tau]$ and $[\tau, t_1]$are admissible, then $u(\cdot)$ is admissible too,

6. if controls $u'(\cdot), u''(\cdot) : [t_0, t_1] \to U$ are admissible, and if $0 < \lambda < 1$, then the control $u(\cdot) : [t_0, t_1] \to U$ defined by $u(t) = (1 - \lambda)u'(t) + \lambda u''(t)$ is admissible too.

The phase space X is the n-dimensional Euclidean space R^n. Let $A \in L(R^n, R^n)$ and $B \in L(R^r, R^n)$. An absolutely continuous function $x(\cdot)[t_0, t_1] \to X$ is a trajectory which corresponds to the admissible control $u(\cdot) : [t_0, t_1] \to U$ if $\dot{x}(t) = Ax(t) + Bu(t)$ a. e. on $[t_0, t_1]$. Let us suppose that in the phase space X

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two convex, closed, disjoint sets X_0 and X_1 are given. We shall call them *initial* and *terminal* set. The trajectory $x(\cdot) : [t_0, t_1] \to X$ accomplishes the passage from X_0 to X_1 if $x(t_0) \in X_0$ and $x(t_1) \in X_1$. The difference $t_1 - t_0$ is then called the *passage time.*

The aim of this paper is to study the problem of minimization of the passage time. The admissible control $\hat{u}(\cdot)$ and the corresponding trajectory $\hat{x}(\cdot)$ are optimal, if trajectory $\hat{x}(\cdot)$ accomplishes the passage from X_0 to X_1 in the shortest time.

In [**2, 5, 3**] and [**1**] some special cases on the mentioned optimal control problem were investigated. Gamkrelidze [**2**] studied the problem with fixed endpoints, namely the case when the control set U is a rectangular parallelepiped and the class of admissible controls D is the class of piecewise continuous controlos. A similar problem was considered in the monograph [**5**] (chapter 3), were the control set U was allowed to be an arbitrary convex polyhedral. In [**3**] the problem with fixed endpoints was studied too, with the control set U being an arbitrary convex compact set, and with measurable admissible controls. The problem with variable endpoints was studied for the first time by Boltyanskii [**1**]. In [**1**], domain of control was an arbitrary convex compact set and admissible controls were piecewise continuous controls.

In Section 2 of the present paper a maximum principle for our problem formulated above is proved. Tha first maximum principle, for any optimal control problem whatever, was proved by Gamkrelidze [**2**]. Maximum principles for versions of the linear optimal control problem in [**5**] and [**3**] were derived from the maximum principle for the general optimal control problem, proved in Chapter 2 of [**5**]. For the linear optimal control problem with variable endpoints, a maximum principle could not be derived from the corresponding theorem for the general problem, because the sets X_0 and X_1 need not have a smooth boundary. Boltyanskiĭ[**1**] proved a maximum principle for the linear optimal control problem with variable endpoints under the additional assumption that the terminal set X_1 is strongly stable. Here we proe a maximum principle without this additional assumption.

In Section 3 two theorems are proved, the first of them being a generalization of Theorem 9 of [**5**] about the finite number of switchings. In the second theorem sufficient conditions for the uniqueness of optimal control are given.

In section 4 an existence theorem for the optimal control problem is proved by refining the reasoning in the proofs of the existence theorems of Gamkrelidze [**2**] and of the monograpf [**5**] (Chapter 3). The crucial role in this proof is played by the proposition about the representation of a closed convex set in Euclidean space. This proposition and its a application in the proofs of existence theorems are presented in [**4**].

At the beginning of Sections 2, 3 and 4 three lemmas are proved, which are used in the theorems which follows. Although these lemmas are known, their proofs are repeated here for the sake of completeness.

The question of sufficient conditions for optimality will not be considered in this paper. The theory developed by Boltyanskiĭ[**1**] concerning sufficient conditions is fully applicable to the problems studied here.

2. Necessary condition for optimality

LEMMA 2.1. *If* $u(\cdot) : [t_0, t_1] \to U$ *is an admissible control,* $x(\cdot) : [t_0, t_1] \to X$ *is the corresponding trajectory and* $p(\cdot) : [t_0, t_1] \to X^*$ *is o solution of the differential equation* $\dot{p} = -pA$, *then*

$$\int_{t_0}^{t_1} p(t)Bu(t)dt = p(t_1)x(t_1) - p(t_0)x(t_0).$$

Proof. The equlities

$$\begin{aligned} \frac{d}{dt}p(t)x(t) =& p(t)x(t) + p(t)\dot{x}(t) \\ =& -p(t)Ax(t) + p(t)Ax(t) + p(t)Bu(t) \\ =& p(t)Bu(t) \end{aligned}$$

hold a. e. on $[t_0, t_1]$. Hence we have

$$\int_{t_0}^{t_1} p(t)Bu(t)dt = p(t)x(t)\Big|_{t_0}^{t_1} = p(t_1)x(t_1) - p(t_0)x(t_0).$$

■

THEOREM 2.1 (*Maximum principle) If* $\hat{u}(\cdot) : [\hat{t}_0, \hat{t}_1] \to U$ *and* $\hat{x}(\cdot) : [\hat{t}_0, \hat{t}_1] \to X$ *are optimal control and corresponding optimal trajectory, there exists a function* $\hat{p}(\cdot) : [\hat{t}_0, \hat{t}_1] \to X^*$, *which is the nontrivial solution of the differential equation* $\dot{p} = -pA$, *such that the following conditions are fulfilled:*

1. *maximum condition:*

$$\max_{u \in U} \hat{p}(t)Bu = \hat{p}(t)B\hat{u}(t) \ \text{ for a. a. } t \in [\hat{t}_0, \hat{t}_1];$$

2. *condition of transversality at the left endpoint:*

$$\max_{x \in X_0} \hat{p}(\hat{t}_0)x = \hat{p}(\hat{t}_0)\hat{x}(\hat{t}_0);$$

3. *condition of transversality at the right endpoint:*

$$\max_{x \in X_1} \hat{p}(\hat{t}_1)x = \hat{p}(\hat{t}_1)\hat{x}(\hat{t}_1).$$

Proof. The sphere of accessibility $S_T, T > 0$, is the set of phase points x_0 from which the passage to the terminal set X_1 can be accomplished in time T. Let us prove that it is convex. Let $x_0', x_0'' \in S_T$ and $x_0 \in]x_0', x_0''[$. There exist two admissible controls $u'(\cdot), u''(\cdot) : [0, T] \to U$ and the corresponding trajectories $x'(\cdot),\ x''(\cdot) : [0, T] \to X$, such that $x'(0) - x_0' = x_0', x''(0) = x_0''$ and $x'(T), x''(T) \in X_1$. There exists a real number $\lambda,\ 0 < \lambda < 1$, such that $x_0 = (1-\lambda)x_0' + \lambda x_0''$. Let us define the functions $u(\cdot) : [t_0, t_1] \to U$ and $x(\cdot) : [0, t] \to X$ by

$$u(t) = (1-\lambda)u'(t) + \lambda u''(t),\ x(t) = (1-\lambda)x'(t) + \lambda x''(t).$$

Here, $u(\cdot)$ is an admissible control and $x(\cdot)$ is the corresponding trajectory. Since

$$x(0) = (1-\lambda)x'(0) + \lambda x''(0) = (1-\lambda)x_0' + \lambda x_0'' = x_0$$
$$x(T) = (1-\lambda)x'(T) + \lambda x''(T) \in X_1,$$

the trajectory $x(\cdot)$ accomplishes the passage from the point x_0 to the terminal set X_1 and therefore $x_0 \in S_T$.

Let $\hat{T} = \hat{t}_1 - \hat{t}_0,\ \hat{x}_0 = \hat{x}(\hat{t}_0)$ and $\hat{x}_1 = \hat{x}(\hat{t}_1)$. Then $\hat{x}_0 \in X_0 \cap S_{\hat{T}}$. The convex sets X_0 and $S_{\hat{T}}$ can be separated by a hyperplane. Let us suppose the contrary. Then relint $X_0\cap$ relint $S_{\hat{T}} \neq \varnothing$. We can suppose that $\hat{x}_0 \in$ relint $X_0\cap$ relint $S_{\hat{T}}$. There exist two planes Y and Z in X such that $Y \subseteq \operatorname{aff} X_0$, $Z \subseteq \operatorname{aff} S_{\hat{T}}$, $Y \cap Z = \{\hat{x}_0\}$ and dim $Y +$ dim $Z =$ dim X. There exists a simplex $\Delta_0 \subseteq Z\cap$ relint $S_{\hat{T}}$, with vertices $x_1, x_2, \ldots, x_{m+1} (m =$ dim $Z)$, such that $\hat{x}_0 \in$ relint Δ_0. For each $i = 1, 2, \ldots, m+1$, there exists an admissible control $u_i(\cdot) : [0, \hat{T}] \to U$ with the corresponding trajectory $x_i(\cdot) : [0, \hat{T}] \to X$ which accomplishes the passage from the point x_i to the terminal set X_1. For sufficiently small $\tau > 0$, points $x_i(\tau), i = 1, 2, \ldots, m+1$, are vertices of the simplex Δ_τ which has a nonempty intersection with the initial set X_0. Because of convexity, the sphere of accessibility $S_{\hat{T}-\tau}$, together with its vertices, contains the whole simplex Δ_τ. Therefore $X_0 \cap S_{\hat{T}\tau} \neq \varnothing$, which contradicts the fact that $\hat{T}$ is the shortest passage time.

There exists a $\hat{p}_0 \in X^*,\ \hat{p}_0 \neq 0$, such that

$$X_0 \subseteq \{x \in X \mid \hat{p}_0 x \leq \hat{p}_0 \hat{x}_0\}, \quad S_{\hat{T}} \subseteq \{x \in X \mid \hat{p}_0 x \geq \hat{p}_0 \hat{x}_0\}.$$

Let $\hat{p}(\cdot) : [\hat{t}_0, \hat{t}_1] \to X^*$ be the solution of the differential equation $\dot{p} = -pA$, which satisfies the condition $\hat{p}(\hat{t}_0) = \hat{p}_0$.

Obviously, the condition of transversality at the left endpoint is fulfilled.

Let $x_1 \in X_1$ and let $x(\cdot) : [\hat{t}_0, \hat{t}_1] \to X$ be the trajectory corresponding to the control $\hat{u}(\cdot)$, which ends at the point x_1. Let us denote by x_0 its initial point. According to Lemma 2.1, we have

$$\hat{p}(\hat{t}_1)x_1 - \hat{p}(\hat{t}_0)x_0 = \hat{p}(\hat{t}_1)\hat{x}_1 - \hat{p}(\hat{t}_0)\hat{x}_0 = \int_{\hat{t}_0}^{\hat{t}_1} \hat{p}(t)B\hat{u}(t)dt.$$

Since $x_0 \in S_{\hat{T}}$, it follows that $\hat{p}(\hat{t}_0)x_0 \geq \hat{p}(\hat{t}_0)\hat{x}_0$. Therefore $\hat{p}(\hat{t}_1)x_1 \geq \hat{p}(\hat{t}_1)\hat{x}_1$. It follows that the condition of transversality at the right endpoint is satisfied.

Let us suppose that the maximum condition is not satisfied.

There exists a $u \in U$ such that $mE > 0$, where

$$E = \{t \in [\hat{t}_0, \hat{t}_1] \mid \hat{p}(t)Bu > \hat{p}(t)B\hat{u}(t)\}.$$

Let us suppose the contrary. Let $\{u_k \mid k \in N\}$ be an everywhere dense set of points in U. Then $mE_k = 0$, where

$$E_k = \{t \in [\hat{t}_0, \hat{t}_1] \mid \hat{p}(t)Bu_k > \hat{p}(t)B\hat{u}(t)\}$$

Since

$$\sup_{k\in N} \hat{p}(t)Bu_k \leq \hat{p}(t)B\hat{u}(t),$$

for $t \in [\hat{t}_0, \hat{t}_1] \setminus \underset{k\in N}{\cup} E_k$, then

$$\max_{u\in U} \hat{p}(t)Bu = \hat{p}(t)B\hat{u}(t),$$

for almost all $t \in [\hat{t}_0, \hat{t}_1]$. Contradiction!

There exists an interval $I \subseteq [\hat{t}_0, \hat{t}_1]$ such that

$$\int_I (\hat{p}(t)Bu - \hat{p}(t)B\hat{u}(t))dt > 0.$$

Let us suppose the contrary. Let

$$\int_E (\hat{p}(t)Bu - \hat{p}(t)B\hat{u}(t))dt = \varepsilon > 0.$$

There exists a $\delta > 0$, such that

$$\Big| \int_\Delta (\hat{p}(t)Bu - \hat{p}(t)B\hat{u}(t))dt \Big| < \varepsilon,$$

for each $\Delta \subseteq [\hat{t}_0, \hat{t}_1]$, $m\Delta < \delta$. Since the set E is measurable, a sequence of disjoint intervals $I_k \subseteq [\hat{t}_0, \hat{t}_1]$, $k \in N$, exists, such that $E \subseteq \underset{k\in N}{\cup} I_k$, $m\Delta < \delta$, where $\Delta = \underset{k\in N}{\cup} I_k \setminus E$. Then

$$\sum_{k=1}^{\infty} \int_{Ik} (\hat{p}(t)Bu - \hat{p}(t)B\hat{u}(t))dt =$$
$$= \int_E (\hat{p}(t)Bu - \hat{p}(t)B\hat{u}(t))dt + \int_\Delta (\hat{p}(t)Bu - \hat{p}(t)B\hat{u}(t))dt > 0.$$

Contradiction!

Let $u(\cdot) : [\hat{t}_0, \hat{t}_1] \to U$ be an admissible control defined by

$$u(t) = \begin{cases} \hat{u}(t), & t \in [\hat{t}_0, \hat{t}_1] \setminus I \\ u, & t \in I \end{cases}$$

and let $x(\cdot) : [\hat{t}_0, \hat{t}_1] \to X$ be the corresponding trajectory which terminates at the point $\hat{x}_1$. According to Lemma 2.1 we have

$$\int_{\hat{t}_0}^{\hat{t}_1} \hat{p}(t)B\hat{u}(t)dt = \hat{p}(\hat{t}_1) - \hat{p}(\hat{t}_0)\hat{x}_0,$$
$$\int_{\hat{t}_0}^{\hat{t}_1} \hat{p}(t)Bu(t)dt = \hat{p}(\hat{t}_1) - \hat{p}(\hat{t}_0)x(\hat{t}_0).$$

It follows that

$$\hat{p}(\hat{t}_0)\hat{x}_0 - \hat{p}(\hat{t}_0)x(\hat{t}_0) = \int_{\hat{t}_0}^{\hat{t}_1} (\hat{p}(t)Bu(t) - \hat{p}(t)B\hat{u}(t))dt$$
$$= \int_I (\hat{p}(t)Bu - \hat{p}(t)B\hat{u}(t))dt > 0.$$

On the other hand, we have $\hat{p}(\hat{t}_0)\hat{x}_0 - \hat{p}(\hat{t}_0)\hat{x}(\hat{t}_0) \leq 0$, because $x(\hat{t}_0) \in S_{\hat{t}}$. Contradiction!

As we see, it turns out that if we suppose that the maximum condition does not hold, we come to a contadiction, and hence, the maximum condition must hold. ■

3. The uniqueness of optimal control

LEMMA 3.1. *Let $p(\cdot) : [t_0, t_1] \to X^*$ be a nontrivial solution of the differential equation $\dot{p} = -pA$ and let Y be a subspace of the phase space X. If $p(t)Y = \{0\}$ for infinitely many $t \in [t_0, t_1]$, then the subspace Y belongs to a proper subspace of X, which is invariant under the operator A.*

Proof. The set of points from the interval $[t_0, t_1]$, for which $p(t)Y = \{0\}$ has at least one accumulatiom point. Let τ be such a point. Let $y \in Y$. Due to the continuity of the function $p(t)y$, the equality $p(\tau)y = 0$ is valid. The derivative of the function $p(t)y$ is given by $\frac{d}{dt}p(t)y = -p(t)Ay$. Since between every two zeroes of a differentiable function lies at least one zero of its derivative, the function $p(t)Ay$ vanishes on an infinite subset of the interval $[t_0, t_1]$, for which τ is an accumulation point. Continuity implies the equality $p(\tau)Ay = 0$. If we proceed with such a reasoning, we can prove that $p(\tau)A^k y = 0$ for every $k \in N$. Since $p(\cdot)$ is a nontrivial solution of a homogeneous linear differential equation, then $p(\tau) \neq 0$. It follows that all vectors $A^k y,\ k = 0, 1, 2, \ldots, y \in Y$, belong to the hypersubspace $H\{x \in X \mid p(\tau\, x = 0\}$. These vectors generate the minimal subspace Z of the phase space X, which is invariant under the operator A and contains Y. As $Z \subseteq H$, we have that Z is a proper subspace of X.■

We shall say that the control set U is in the general position if there exists a nonempty countable family $\mathcal{S}$ of subspaces of the space R^r, which satisfies the following two conditions:

1. each hyperplane of support of the set U, which has more than one common point with U, is parallel to some subspace from $\mathcal{S}$,

2. if Z is a subspace from $\mathcal{S}$, then BZ is not contained in any proper subspace of X which is invariant under the operator A.

THEOREM 3.1. *Let the control set U be in the general position. If $p(\cdot) : [t_0, t_1] \to X^*$ is a nontrivial solution of the differential equation $\dot{p} = -pA$, then the control $u(\cdot) : [t_0, t_1] \to U$ is uniquely determined by the maximum condition*

$$\max_{u \in U} p(t)Bu = p(t)Bu(t)$$

in all but countably many points of the interval $[t_0, t_1]$ *The control* $u(\cdot)$ *is continuous at every point at which it is uniquely determined.*

Proof. Let $Z \in \mathcal{S}$. According to the preseding lemma, the equality $p(t)BZ = \{0\}$ is fulfilled for finitely many $t \in [t_0, t_1]$. So, the set of points $t \in [t_0, t_1]$, for which there exists a $Z \in \mathcal{S}$ such that $p(t)BZ = \{0\}$, is countable. Let $\tau \in [t_0, t_1]$ be a point such that $p(\tau)BZ \neq \{0\}$, for each $Z \in \mathcal{S}$. The function $u \to p(\tau)Bu$ reaches its maximum on the set U at the unique point $u(\tau)$. Let $\varepsilon > 0$. The function

$$u \to \frac{p(\tau)B(u - u(\tau))}{\|u - u(\tau)\|}$$

is continous and negative on the compact set $U \backslash B]u(\tau), \varepsilon[$. Therefore a $\mu > 0$ exists, such that

$$p(\tau)B(u - u(\tau)) \leq -\mu\|u - u(\tau)\|,$$

for every $u \in U \backslash B]u(\tau),\ \varepsilon[$. There exists a $\delta > 0$, such that

$$\|p(t) - p\tau)\| < \mu/\|B\|$$

when $\mid t - \tau \mid < \delta$. Let $\mid t - \tau \mid < \delta$. For $u \in U \backslash B]u(\tau),\ \varepsilon[$ we have

$$\begin{aligned} p(t)Bu &= (p(t) - p(\tau))B(u - u(\tau)) + p(\tau)B(u - u(\tau)) + p(t)Bu(\tau) \\ &< \mu\|u - u(\tau)\| - \mu\|u - u(\tau)\| + p(t)Bu(\tau) \\ &= p(t)Bu(\tau). \end{aligned}$$

Since $u(t)$ is the point at which the function $u \to p(t)u$ reaches its maximum on the set U, then $u(t) \in B]u(\tau), \varepsilon[$.$\blacksquare$.

Remark. If the family $\mathcal{S}$ is finite, then we can conclude that the set of points at which the control $u(\cdot)$ is not uniquely determined is finite too.

Theorem 3.2.. *Let the control set* U *be in the general position. Then each two optimal controls defined on the same interval coincide.*

Proof. Let $\hat{u}(\cdot) : [\hat{t}_0, \hat{t}_1] \to U$ and $\hat{x}(\cdot) : [\hat{t}_0, \hat{t}_1] \to X^*$ be the optimal control and the corresponding optimal trajectory. Let $\hat{p}(\cdot) : [\hat{t}_0, \hat{t}_1] \to X^*$ be the nontrivial solution of the differential equation $\dot{p} = -pA$, such that conditions 1,2 and 3 of Theorem 2.1 are fulfilled. Let $\hat{u}(\cdot) : [\hat{t}_0, \hat{t}_1] \to U$ and $\hat{x}(\cdot) : [\hat{t}_0, \hat{t}_1] \to X$ be another optimal control and the corresponding optimal trajectory. Since

$$\begin{gathered} \hat{p}(\hat{t}_0)\hat{x}(\hat{t}_0) \geq \hat{p}(\hat{t}_0)x(\hat{t}_0)\ \hat{p}(\hat{t}_1)\hat{x}(\hat{t}_1) \leq \hat{p}(\hat{t}_1)x(\hat{t}_1), \\ \int_{\hat{t}_0}^{\hat{t}_1} \hat{p}(t)B\hat{u}(t)dt = \hat{p}(\hat{t}_1)\hat{x}(\hat{t}_1) - \hat{p}(\hat{t}_0)\hat{x}(\hat{t}_0), \\ \int_{\hat{t}_0}^{\hat{t}_1} \hat{p}(t)Bu(t)dt = \hat{p}(\hat{t}_1)\hat{x}(\hat{t}_1) - \hat{p}(\hat{t}_0)\hat{x}(\hat{t}_0), \end{gathered}$$

then

$$\int_{\hat{t}_0}^{\hat{t}_1} \hat{p}(t)B\hat{u}(t)dt = \leq \int_{\hat{t}_0}^{\hat{t}_1} \hat{p}(t)Bu(t)dt$$

Besides, $\hat{p}(t)B\hat{u}(t) \geq \hat{p}(t)Bu(t)$ a.e. on $[\hat{t}_0, \hat{t}_1]$. It follows that

$$\max_{u \in U} \hat{p}(t)Bu - \hat{p}(t)B\hat{u}(t) = \hat{p}(t)Bu(t)$$

a.e. on $[\hat{t}_0, \hat{t}_1]$. According to the previous theorem, we conclude that $u(t)\hat{u}(t)$ a.e. on $[t_0, t_1]$.■.

4. Existence of optimal control

LEMMA 4.1 *Every closed convex set C in Euclidien space X can be represented as an intersection of a coutable family of closed half- spaces.*

Proof. Let $\{x_k \mid \in N\}$ be an everywhere dense set of points in aff $C \backslash C$. According to a well-known theorem, for each $k \in N$, there exists a closed half-space P_k such that $C \subseteq P_k$ and $x_k \notin P_k$. Let us prove that $C = (\text{agg} C) \cap \bigcap_{k \in N} P_k$. Obviously, $C \subseteq (\text{aff} C) \cap \bigcap_{k \in N} P_k$. Let us suppose that $x \notin C$. If $x \notin \text{aff} C$, it is clear that $x \notin (\text{ aff } C) \cap \bigcup_{k \in N} P_k$. Let $x \in (\text{aff} C) \backslash C$. Let us denote by $\tilde{C}$ the convex hull of the set $C \cup \{x\}$. Let $a \in$ relint C and let b be the intersection of the segment $]a, x[$ and relbd C. The set (relint $\tilde{C}$)$\backslash C$ is nonempty, since it contains the segment $]b, x[$. An integer $k \in N$ exists, such that $x_k \in$ (relint $\tilde{C}$)$\backslash C$. We have that $x \notin P_k$. Otherwise we would have $x_k \in \tilde{C} \subseteq P_k$. It follows that $x \notin (\text{aff } C) \cap \bigcap_{k \in N} P_k$. Thus we have proved the equality $C = (\text{ aff } C) \cap \bigcap_{k \in N} P_k$. It remains only to note that aff C can be represented as the intersection of a finite family closed half-spaces. ■.

THEOREM 4.1. *Let the class D of admissible controls be maximal, i.e. let D be the class of all measurable controls, and let one of sets X_0 and X_1 be compact. If there exists at least one admissible control with the corresponding trajectory which accomplishes the passage from the set X_0 to the set X_1 then the optimal control exist.*

Proof. We can suppose that the initial set X_0 is compact.

Let $\hat{T}$ be the infimum over all passage times from the set X_0 to the set X_1. There exists a sequence of admissible controls $u_k(\cdot)[0, T_k] \to U$ with corresponding trajectories $x_k(\cdot) : [0, T_k] \to X$ accomplishing the passage from X_0 to X_1, such that $T_k \to \tilde{T}$. Corresponding trajectories are given by

$$x_k(t) = \Phi(t) \left[x_k(0) + \int_0^t \Phi(\tau)^{-1} Bu_k(\tau) d\tau \right],$$

where $\Phi(\cdot) : R \to L(R^n, R^)$ is the solution of the differetial equation $\Phi = A \cdot \Phi$ satisfying the initial condition $\Phi(0) = I$.

Sequences $x_k(0)$ and $x_k(T_k)$ are bounded. We can suppose that they are convergent. Let $x_k(0) \to \hat{x}_0$ and $x_k(T_k) \to \hat{x}_1,\ k \to \infty$. Obviously $\hat{x}_0 \in X_0$ and $\hat{x}_1 \in X_1$. Since

$$\begin{aligned} x_k(\hat{T}) - \hat{x}_1 =& [x_k(\hat{T}) - x_k(T_k)] + [x_k(T_k) - \hat{x}_1] \\ =& [\Phi(\hat{T}) - \Phi(T_k)]\left[x_k(0) + \int_0^{\hat{T}} \Phi(\tau^{-1} B u_k(\tau) d\tau\right] \\ & - \Phi(T_k) \int_{\hat{T}}^{T_k} \Phi(\tau)^{-1} B u_k(\tau) d\tau + [x_k(T_k - \hat{x}_1] \end{aligned}$$

we have $x_k(\hat{T}) \to \hat{x}_1,\ k \to \infty$. This implies $\hat{T} > 0$.

If we consider the sequence of controls $(u_k(\cdot))$ as a sequence of points in the space $L_2^r[0, \hat{T}]$, then it is a bounded sequence. It has a weakly convergent subsequence. We may assume without loss of generality that the sequence $(u_k(\cdot))$ is weakly convergent. Let $u_k(\cdot) \to \hat{u}(\cdot),\ k \to \infty$. As $\hat{u}(\cdot)$ is a point in the space $L_2^r[0, \hat{T}]$, it is measurable function which maps $[0, \hat{T}]$ into R^r.

Let us prove that $\hat{u}(t) \in U$ for almost all $t \in [0, \hat{T}]$. As, according to the previous lemma, U is an intersection of a countable family of half-spaces, it suffices to prove that for each closed half-space $P \supseteq U$ we have $\hat{u}(t) \in P$ a. e. on $[0, \hat{T}]$. The half-space P can be represented in the form $P = \{u \in R^r \mid au \leq \alpha\}$, where $a \in X^*$ and $\alpha \in R$. Let $E_\lambda = \{t \in [0, \hat{T}] \mid a\hat{u}(t) \geq \lambda\}$, for $\lambda \in R$. Since

$$\int_0^{\hat{T}} K_{E_\lambda}(t) a u_k(t) dt \to \int_0^{\hat{T}} K_{E_\lambda}(t) a \hat{u}(t) dt,\ \ k \to \infty,\ \text{ and}$$

$$\int_0^{\hat{T}} K_{E_\lambda}(t) a u_k(t) dt \to \int_{E_\lambda} a u_k(t) dt \leq \alpha m E_\lambda,\ \ k \in N,$$

we have

$$\int_0^{\hat{T}} K_{E_\lambda}(t) a \hat{u}(t) dt \leq \alpha m E_\lambda$$

On the other hand,

$$\int_0^{\hat{T}} K_{E_\lambda}(t) a \hat{u}(t) dt = \int_{E_\lambda} a \hat{u}(t) dt \geq \lambda m E_\lambda.$$

Hence $mE_\lambda = 0$ for every $\lambda > \alpha$. Since the relation $\hat{u}(t) \in P$ is not valid only on the set $\bigcap_{s \in N} E_{\alpha + 1/s}$, we conclude that it is valid a.e. on $[0, \hat{T}]$.

The change of the values of the function $\hat{u}(\cdot)$ on a set of measure zero does not affect the weak convergence of the sequence $(\hat{u}_k(\cdot))$ to $u(\cdot)$. Therefore, we may assume that $u(t) \in U$, for each $t \in [0, \hat{T}]$, so that $\hat{u}(\cdot)$ i an admissible control.

Let $\hat{x}(\cdot) : [0, \hat{T}] \to X$ be the trajectroy corresponding to the admissible control $\tilde{u}(\cdot)$, satisfying the initial condition $\hat{x}(0) = \hat{x}_0$. It can be represented in the form

$$\hat{x}(t) = \Phi(t)\left[\hat{x}_0 + \int_0^t \Phi(\tau)^{-1} B \hat{u}(\tau) d\tau\right].$$

Since

$$\int_0^{\hat{T}} \Phi(\tau^{-1} B u_k(\tau) d\tau \to \int_0^{\hat{T}} \Phi(\tau)^{-1} B \hat{u}(\tau) \delta\tau, \quad k \to \infty$$

we have $x_k(\hat{T}) \to \hat{x}(\hat{T})$, as $k \to \infty$. On the other hand, $x_k(\hat{T}) \to \hat{x}_1$, as $k \to \infty$. It follows that $\hat{x}(\hat{T}) = \hat{x}_1$.■.

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