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CHEBYSHEV CENTRES IN NORMED SPACES

Lazar Pevac

Abstract. The existence of Chebyshev centres and best compact approximants supposing special geometrical properties for the normed space in investigated. The positive results are obtained using a slightly changed of quisi-uniform convexity noted in [**1**]

Let X be a normed space, A and B bounded subset of X and x an element of X. Let us denote

$$d(x,A)=\inf_{Y\in A}\|x-y\|,\ K(A,r)=\{x\in X\mid d(x,A)\le r\},$$
$$\partial(B,A)=\inf\{r\ge 0\mid B\subset K(A,r)\}.$$

The number $R(A)=\inf\{\partial(A,x)\mid x\in X\}$ is called *Chebyshev radius* of A and the set $(C(A)=\{x^*\in X\mid \partial(A,x^*)\}$ is called *Chebyshev centre* set of A. We say that X *admits centre* if for every bounded set A of $X, C(A)\neq\varnothing$.

If $\mathcal{K}$ is the family of all compact subset of X then number $R_{\mathcal{K}}(A)=\inf_{K\in\mathcal{K}}\partial(A,K)$ is a *compact radius* of A. If there exists a $K^*\in\mathcal{K}$ such that $\partial(K^*,A)=R_{\mathcal{K}}(A)$ then we say that set A has the *best compact approximant.*

Definition 1. We say that the normed space X is α-approximative iff $\forall\varepsilon(0<\varepsilon<1)\exists(\delta)(\varepsilon)$ tending to 0 when ε tends to 0 and $0\le\delta(\varepsilon)<1$ such that $\forall x\in X\ \exists y\in X$ with $\|y\|\le\delta(\varepsilon)$, and such that if $z\in X$ and

$$\|z\|\le 1 \text{ and } \|z-x\|\le 1-\varepsilon$$

then also

$$\|z-y\|\le 1-\varepsilon(1-\alpha).$$

The definition above has the following geometrical meaning. For every ball $K(x,\ 1-\varepsilon)$ heaving nonempty intersection with the unit ball $K(0,1)$, there exists a ball $K(y,1-\varepsilon(1-\alpha))$ containg $K(x,1-\varepsilon)\cap K(0,1)$ so that the centre y of

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$K(y, 1-\varepsilon(1-\alpha))$ is not "so far away" from the origin, i. e. y is contained in $K(0, \delta(\varepsilon))$. It is represented by Fig. 1.

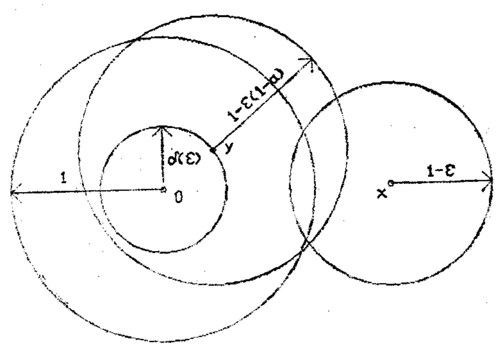

Fig. 1.

COROLLARY 1.

(i) If $\alpha \geq 1$ then every normed space is α-approximative.

(ii) If $\alpha < 0$ then there is no normed space which would be α-approximative.

(iii) If X is α-approximative and $\delta(\varepsilon)$ is not decreasing, we can replace the function $\delta(\varepsilon)$ with a decreasing function so that X remains α-approximative.

(iv) $\delta(\varepsilon) \geq \varepsilon(1-\alpha)$.

(v) If X is α-approximative and $r > 0$ then $\forall \varepsilon (0 < \varepsilon < r) \exists \delta_1(\varepsilon)$ tending to 0 when ε tends to 0, and $0 < \delta_1(\varepsilon < r$ such that $\forall x \in X \ \exists y \in X$ with $\|y\| \leq \delta_1(\varepsilon)$, and such that if $z \in X$ and

$$\|z\| \leq r \text{ and } \|z-x\| \leq r-\varepsilon \text{ then also } \|z-y\| \leq r-\varepsilon(1-\alpha)$$

where $\delta_1(\varepsilon) = r\delta(\varepsilon/r)$.

(vi) If X is α-approximative and $0 < R_1 \leq r \leq R_2$ then $\forall \varepsilon (0 < \varepsilon < R_1) \exists \delta_2(\varepsilon)$ tending to 0 when ε tends to 0 and $0 < \delta_2(\varepsilon) < R_2$ such that $\forall x \in X \ \exists y \in X$ with $\|y\|\delta_2(\varepsilon)$, and such that if $z \in X$ and

$$\|z\| \leq r \text{ and } \|z-x\| \leq r-\varepsilon \text{ then also } \|z-y\| \leq r-\varepsilon(1-\alpha)$$

where $\delta_2(\varepsilon) = R_2\delta(\varepsilon/R_1)$.

Proof. The properties (i) and (ii) suggest that it is not interesting to consider the cases $\alpha \geq 1$ or $\alpha < 0$. The proof of the properties is obvious. The property (iii) suggests that we always may assume that $\delta(\varepsilon)$ is decreasing, without loss of generality. Suppose that $\delta(\varepsilon)$ is not decreasing. Let us consider the function:

$$\delta_1(\varepsilon) = \sum_n x_{J_n} \sup_{I_n} \delta(\varepsilon)$$

where $J_n = [\varepsilon_{n+1}, \varepsilon_k]$, $I_n = [0, \varepsilon_n]$ and x_{J_n} is the characteristic function of J_n, and finally, (ε_n) is a sequence decreasing to 0. As $\delta_1(\varepsilon) \geq \delta(\varepsilon)$, X remains α-approximative when we replace $\delta(\varepsilon)$ with $\delta_1(\varepsilon)$. So the property (iii) is proved.

For proving the property (iv), we shall choose z from the definition 1, such that $\|z\| = 1$. When, further, we apply the triangle rule to the elements z, y and $z - y$, we have $\|z\| \leq \|y\| + \|z - y\|$. So we get $1 \leq \delta(\varepsilon) + 1 - \varepsilon(1 - \alpha)$, whence $\delta(\varepsilon) \geq \varepsilon(1 - \alpha)$ and (iv) is proved.

We have to map the given balls $K(0, r)$ and $K(x, r - \varepsilon)$ homoteticaly with factor $1/r$. So we get balls $K(0, 1)$ and $K(x/r, 1 - \varepsilon/r)$. Applying the definition we get the element $y \in K(0, \delta(\varepsilon/r))$. With the inverse homotetical map we are going back to the starting position. Thus the element $y_1 = ry$ is contained in $K(0, r\delta(\varepsilon/r))$ which proves (v). The property (vi) immediately follows form the (iii) and (v) since for every $r, R_1 \leq r \leq R_2$ we have $r\delta(\varepsilon/r) \leq R_2\delta(\varepsilon/R_1)$.

Corollary 2. *If the space X is uniformly convex then X is $0-$approximative.*

Proof. From uniform convexity of the space X it is easy to show that $\forall \varepsilon < 0\, \exists \eta(\varepsilon)$ such that if $x, y \in X$ and $\|x\| \leq 1$ and $\|y\| \leq 1$ and $\|x - y\| \geq \varepsilon$ then also $\|(x - y)/2\| \leq 1 - \eta(\varepsilon)$. When we replace x and y by $x - z$ and $y - z$, respectively, where z is an arbitrary element form X, we obtain

$$\|x - z\| \leq 1 \wedge \|y - z\| \leq 1 \wedge \|x - y\| \geq \varepsilon \Rightarrow \|(x - y)/2 - z\| \leq 1 - \eta(\varepsilon).$$

The relation above has the simple geometrical meaning. If $K(x, 1)$ and $K(y, 1)$ are balls in X, having nonempty intersection, and $\|x - y\| \geq \varepsilon$, then $K(x, 1) \cap K(y, 1)$ is contained in the ball whose centre is the midpoint between x and y and whose radius is equal to $1 - \eta(\varepsilon)$. Let X, be uniformly convex and $K(0, 1)$ and let $K(x, 1 - \varepsilon)$ are given balls in X. Put

$$\delta_1(\varepsilon) = \min\{\sigma \mid (0, 1) \cup K(x, 1 - \varepsilon) \subset K(\sigma x/\|x\|, 1 - \varepsilon)\}.$$

The $\delta_1(\varepsilon)$ is well defined because $\sigma\|x\|$ is contained in the set at the right-hand side. Let us prove that $\delta_1(\varepsilon)$ tends to zero when ε tends to zero. On the contrary, let us suppose that $\delta_1(\varepsilon) \to \delta_0 > 0$. Then we can choose ε such that

$$1 - \varepsilon > 1 - \eta(\delta_0). \tag{1}$$

Thus we have $K(0, 1) \cup K(x, 1 - \varepsilon) \subset K(\delta_1(\varepsilon)x/\|x\|, 1 - \varepsilon)$ and $K(\delta_1(\varepsilon)x/\|x\|, 1 - \varepsilon) \subset K(\delta_1(\varepsilon)x/\|x\|, 1) \cup K(0, 1)$. Now we apply the geometrical consequence of uniform convexity, noted before, to balls on the right-hand side of the last relation: $K(0, 1) \cup K(x, 1 - \varepsilon) \subset K(\delta_1(\varepsilon)x/\|2x\|,\ 1 - \delta(\delta_(\varepsilon)))$. Taking into account the inequality (1) we get $K(0, 1) \cup K(x, 1 - \varepsilon) \subset K(\delta_1(\varepsilon)x/\|2x\|, 1 - \varepsilon)$. Using the definition 1. we finally obtain $\delta_1(\varepsilon)/2 \geq \delta_1(\varepsilon)$ which is the contradiction.

Examples. We shall mention some examples of different degrees of approximativity.

The space $\mathcal{C}[0, 1]$ of continous functions on $[0, 1]$ is 0-approximative with $\delta(\varepsilon) = \varepsilon$.

Let us consider the space R^3 with the norm $\|(x, y, z)\| = (x^2 + y^2 + z^2)^{1/2}$. This space is 0-approximative with $\delta(\varepsilon) = (2\varepsilon)^{1/2}$.

If we define norm $\|(x,y,z)\| = | x | + | y | + | z |$ then for some ε and

$$\|(x,y,z)\| = 1, \quad ,\text{no } \|(x_1,y_1,z_1)\| < 1 \text{ satisfy}$$
$$K(0,1) \cap K((x,y,z), 1-\varepsilon) \subset K((x_1,y_1,z_1), 1-\varepsilon).$$

Consequently R^3 cannot be 0-approximative. It is easy to see that in this case R^3 is 0.5-approximative with $\delta(\varepsilon) \leq 3\varepsilon$.

The space l_1 of all absolutely convergent series is not α-approximative for any $\alpha < 1$.

Theorem. *Let X be an α-approximative Banach space with $0 \leq \alpha < 1$. If the series $\sum_{n=1}^{\infty} \delta(\alpha^n)$ converges, then*

(a) every bounded set M in X has a Chebyshev centre,

(b) every bounded set M in X has a best compact approximant.

Proof. Let $R_1(M) = r_0$ and $K(x_n, r_n)$ be the sequence of balls containing M so that r_n decreasing and tends to r_0. The case $r_0 = 0$ is not of interest. On the other hand we can suppose that r_n is less than the diameter of M. We construct the sequence $K(y_n, \rho_n)$ inductively. $K(y_1, \rho_1) = K(x_1, r_1)$.

Suppose that we already have the ball $K(y_{n-1}, \rho_{n-1})$, where $n \geq 2$. Applying corollary 2. (vi) to $K(y_{n-1}, \rho_{n-1})$ and $K(x_n, r_n)$ we get $K(y_n, \rho_n)$ so that

$$\rho_n = r_n + \alpha(\rho_{n-1} - r_n) = \alpha\rho_{n-1} + (1-\alpha)r_n, \; d(y_n, y_{n-1}) \leq \delta(\rho_{n-1} - r_n).$$

After solving the system of the difference equalities we get

$$\rho_n = r_n + \alpha\varepsilon_{n-1} + \alpha^2\varepsilon_{n-2} + \cdots + \alpha^{n-1}\varepsilon_1,$$
$$d(y_n, y_{n-1}) \leq \delta(\varepsilon_n + \alpha\varepsilon_{n-1} + \alpha^2\varepsilon_{n-1} + \cdots + \alpha^{n-1}\varepsilon_1).$$

where $\varepsilon_n = r_{n-1} - r_n$. If $\alpha = 0$ then $\rho_n = r_n$ and $d(y_n, y_{n-1}) = \delta(\varepsilon_n)$. When we choose ε_n so that $\delta(\varepsilon_k) \leq 1/2^n$, the sequence (y_n) converges. If $0 < \alpha < 1$, then we choose the starting sequence $K(x_n, r_n)$ so that $\varepsilon_n < \alpha^{2n}$. Then ρ_n obviously converges to 0, and moreover, $d(y_n, y_{n-1} \leq \delta(\alpha^{n+1}(\alpha^n - 1)/(\alpha - 1))$. Therefore, because of $0 < \alpha < 1$ there exists a integer k so that $\alpha^k/(1-\alpha) < 1$. Consequently $d(y_n, y_{n-1}) \leq \delta(\alpha^{n+1-k})$, and so we conclude that (y_n) converges to the Chebyshev centre of M.

In order to prove the second part of the theorem, we suppose that $R_{\mathcal{K}}(M) = r_0$. Then there exists a real sequence (r_n) tending to r_0, and sequence of nets $(N_n \mid N_n \subset X \wedge \partial(M, N_n) = r_n \wedge \mathit{card}(N_n) < \infty)$. If $r_0 = 0$ then $\mathit{cl}(M)$ is the best compact approximant of M. If r_0 is different from 0, then we will repeat a procedure similar to the proof of the first part of the theorem. Naimelly, we construct the sequences (ρ_n) and (K_n) as follows.

Let $\rho_1 = r_1$ and $K_1 = N_1$. Suppose that the members of sequences of indices less than n are already done. Let us consider the pairs $(x, y) \in N_n \times M_{n-1}$ such that $K(x, r_n) \cap K(y, \rho_{n-1}) \cap M \neq \varnothing$. Applying the corollary 2 (vi) to balls noted above, we get the set K_n such that.

$$\partial(M, K_n) = \rho_n, \quad \rho_n = r_n + \alpha(\rho_{n-1} - r_n) = \alpha\rho_{n-1} + (1-\alpha)r_n$$
$$\partial(K_{n-1}, K_n) \leq \delta(\rho_{n-1} - r_n), \quad card(K_n) \leq card(N_n) card(K_{n-1}).$$

Finally, when we chose the starting sequences so that $\varepsilon_n < \alpha^{2n}$ we get $\partial(K_{n-1}, K_n) \leq \delta(\alpha^{n+1-k})$. Then the set $K = \cup_n K_n$ is totally bounded, hence $cl(K)$ is a compact set. As ρ_n tends to r_0 we have $\partial(M, cl(K)) = r_0$ and the second part of theorem is proved.

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Arhitektonski fakultet
Bulevar Revolucije 73
11000 Beograd
Yugoslavia

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