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EXISTENCE AND UNIQUENESS THEOREM FOR SINGULAR INITIAL VALUE PROBLEMS AND APPLICATIONS

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Abstract. We prove the existence and the uniqueness of the solution to the nonlinear singular initial value problem (1.1) below, and show that such a solution continuously depends on the initial condition. This result is then applied to radially symmetric nonlinear Dirichlet problems.

1. Introduction. We consider the singular initial value problem

$$\begin{aligned} v'' + nt^{-1}v' + g(v) &= p(t) \qquad t \in [0,T] \\ v(0) = d, &\qquad v'(0) = 0 \end{aligned} \tag{1.1}$$

where $n \in N$, $g : R \to R$ is a locally Lipschitzian function, $T > 0$, $p \in L^\infty[0,T]$ and $d \in R$.

Our main result is Theorem 2.1. below, that shows the existence and the uniqueness of the solution to the problem (1.1) on the interval $[0,T]$ continuously depending on the initial condition d.

The result plays a central role in the study of radially symmetric solutions to nonlinear Dirichlet problems, since a "radially symmetric setting" reduces the N-dimensional problem to a one dimensional. In section 3 we show how radially symmetric solutions for superlinear Dirichlet problems, and superlinear Dirichlet problems with jumping can be obtained by combining Theorem 2.1 and the shooting method. For more details and further applications the reader is refered to [**1**] and [**2**].

2. Main Results. In this section we prove:

THEOREM 2.1. *Let a real number $b > 0$ be such that $G(v) := \int_0^v g(u)du \geq 0$ for any $v \in R$, $|v| > b$. If g is locally Lipschitzian, then for every $d \in R$ problem*

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(1.1) *has a unique solution* $v(t,d)$ *on the interval* $[0,T]$ *continuously depending on* d.

Based on the contraction mapping arguments the following proposition gives the continuous dependence of the solution on parameters.

PROPOSITION 2.2. *Let* (X,d) *be a complete metric space,* (Y,δ) *a metric space and* $S: X\times Y\to X$ *a function continuous in the second variable. If there exists a real number* $k\in(0,1)$ *such that* $d(S(x,y),\, S(x',y))\le kd(x,x')$, *for all* $x,x'\in X$, $y\in Y$, *then there exists a continuous function* $f: Y\to X$ *such that* $S(f(y),y)=f(y)$.

Proof. By the contraction mapping principle (see [**3, 4**]) for every $y\in Y$ there exists a unique $x=f(y)$ such that $S(f(y),y)=f(y)$. Now we will prove that the function f is continuous.

Let $\{y_n\}$ be a sequence in Y converging to y. From

$$\begin{aligned} d(f(y),f(y_n)) =&d(S(f(y),y),\ S(f(y_n),y_n))\\ \le& d(S(f(y),y),\ S(f(y),y_n))+d(S(f(y),y_n),\ S(f(y_n),y_n))\\ \le& d(S(f(y),y),\ S(f(y),y_n))+kd(f(y),f(y_n)),\end{aligned}$$

we obtain

$$(1-k)d(f(y),f(y_n))\le d(S(f(y),y),\ S(f(y),y_n)) \tag{2.1}$$

Since S is a continuous function in the second variable, we see from (2.1) that if $y_n\to y$ then $d(S(f(y),y),\ S(f(y),y_n))\to 0$. Hence, $f(y_n)\to f(y)$. Thus f is continuous, and this concludes the proof of the proposition.

The existence and the uniqueness theorem for problem (1.1) heavily uses the fact that the function g is locally Lipschitzian.

Definition. A function $g: R\to R$ is locally Lipschitzian if given any bounded set $R_0\subset R$, there exists a real number $a>0$, depending on R_0, such that for all $r_1,r_2\in R_0\subset R$.

$$\mid g(r_1)-g(r_2)\mid\le a\mid r_1-r_2\mid . \tag{2.2}$$

Let now $d_1\in R$ and let g be a locally Lipschitizian function with a Lipchitz constant a, corresponding to the interval $[d_1-1,d_1+1]$. We define

$$\varepsilon:=\min\{(1/\sqrt{a}),\varepsilon_1\}, \tag{2.3}$$

where ε_1 is such that $(\|p\|_\infty+\mid g(d)\mid +a)\varepsilon_1^2<n$, and d, p and n are as in Theorem 2.1.

THEOREM 2.3. *If* $g: R\to R$ *is locally Lipschitzian, then there exists* $\varepsilon>0$ *such that the problem* (1.1) *has a unique solution* $v(t,d)$ *on* $[0,\varepsilon)\times[d_1-1/4,\ d_1+1/4]$. *Moreover,* $v(\cdot,d)$ *continuously depends on* d *in the topology of* $C([0,\varepsilon),R)$, *defined by the norm* $\|x\|_*=\sup\limits_{t\in[0,\varepsilon)}\mid x(t)\mid$.

Proof. Let $d_1 \in R$, and ε be as in (2.3). We define

$$X := \{v : [0,\varepsilon) \to R : | v(t) - d_1 | \leq 1 \text{ for all } t \in [0,\varepsilon)\},$$

and

$$Y := [d_1 - 1/4, d_1 + 1/4].$$

Now we define the operator $S : X \times Y \to X$ in the following way

$$S(v(t), d) = d + \int_0^t s^{-n} \int_0^t (p(r) - g(v(r)))r^n dr ds. \tag{2.4}$$

We see that S is well defined since

$$\begin{aligned}
\|S(v,d) - d_1\|_* &= \sup_{t\in[0,\varepsilon)} | S(v(t),d) - d_1 | \\
&\leq \sup_{t\in[0,\varepsilon)} \int_0^t s^{-n} \int_0^s | p(r) - g(v(r)) | r^n dr ds + | d - d_1 | \\
&\leq \sup_{t\in[0,\varepsilon)} \int_0^t s^{-n} \int_0^s | p(r) - g(d) | r^n dr ds \\
&+ \sup_{t\in[0,\varepsilon)} \int_0^t s^{-n} \int_0^s | g(d) - g(v(r)) | r^n dr ds + 1/2 \\
&\leq [(\|p\|_\infty + | g(d) |)\varepsilon^2 + a\|v - d\|_* \varepsilon^2]/(2n+2) = 1/2 \\
&\leq [(\|p\|_\infty + | g(d) | + a)/(2n+2)]\varepsilon^2 + 1/2 < 1,
\end{aligned}$$

(see (2.3)). Furthermore, for $v_1, v_2 \in X$ we have

$$\begin{aligned}
\|S(v_1,d) - S(v_2,d)\|_* &= \sup_{t\in[0,\varepsilon)} | S(v_1(t),d) - S(v_2(t),d | \\
&\leq \sup_{t\in[0,\varepsilon)} \int_0^t s^{-n} \int_0^s | g(v_1(r)) - g(v_2(r)) | r^n dr ds \\
&\leq \|g(v_1) - g(v_2)\|_* \frac{\varepsilon^2}{2n+2} \leq \frac{a\varepsilon^2}{2n+2} \|v_1 - v_2\|_* \\
&\leq \frac{1}{2n+2} \|v_1 - v_2\|_* < \frac{1}{2} \|v_1 - v_2\|_*.
\end{aligned}$$

Thus, for all d and for $t \in [0,\varepsilon) S$ is a contraction in v. Hence, since X is a complete metric space by the contraction mapping principle (see Proposition 2.2) for every d there exists a unique solution $v(t,d)$ on $[0,\varepsilon)$ such that $S(v,d) = v(\cdot,d)$, and $v(\cdot,d)$ is a continuous function of d. Thus, Theorem 2.3. is proved.

In order to establish whether the solution $v(t,d)$ can be extended past ε or not, it is necessary to consider its behaviour near ε.

The following lemma roughly says that if a solution cannot be extended any further than the interval $[0,\varepsilon)$, then there is a "blow up", which means that the solution $v(t,d)$ becomes unbounded as $t \to \varepsilon$.

Lemma 2.4. *Let $v(t,d)$ be a solution to* (1.1) *on* $[\alpha,\beta)$. *If* $\lim_{t\to\beta}\sup \mid v(t,d) \mid < \infty$. *then there exists* $\bar{\varepsilon} > 0$ *and* $\bar{v} : [\alpha, \beta+\bar{\varepsilon}) \to R$ *that satisfies* (1.1) *and* $\bar{v} \equiv v$ *on* $[\alpha,\beta)$.

Proof. Let $t_n \to \beta$. Since $\sup \mid v(t,d) \mid < \infty$ and g is a continuous function we see that there exists $M > 0$ such that

$$\mid v(t_n) - v(t_m) \mid \leq \int_{t_n}^{t_m} s^{-n} \int_0^s r^n (p(r) - g(v(r))) dr ds \leq M(t_m - t_n).$$

Hence $\{v(t_n)\}$ is a Cauchy sequence. Thus if we define $v(\beta) = \lim_{n\to\infty} v(t_n) \in R$, then by Theorem 2.3. there exists $\bar{\varepsilon} > 0$ such that there exists a unique solution on $[\beta, \beta+\bar{\varepsilon})$. Moreover, we see that there exists $\bar{v} : [\alpha, \beta+\bar{\varepsilon})$ such that $\bar{v}$ satisfies (1.1) and $\bar{v} \equiv v$ on $[\alpha,\beta)$; thus the lemma is proven.

Proof of Theorem 2.1. From Theorem 2.3. and Lemma 2.4. we see that a solution to (1.1) exists and that it can either be extended to the interval $[0,T]$ or it blows up. Suppose that it blows up, i.e. that there exists $\bar{t} \in [0,T]$ and an increasing sequence $\{t_n\}$ such that $t_n \to \bar{t}$ and $v(t_n,d)^2 \to \infty$. If $(v'(t_n,d))^2$ does not tend to infinity, then by the mean value theorem a new increasing sequence $\{t'_n\}$ exists such that $t'_n \to \bar{t}$ and $v'(t'_n,d) \to \infty$. Thus, without loss of generality, we can assume that

$$(v'(t_n,d))^2 \to \infty \quad \text{as} \quad t_n \to \bar{t}. \tag{2.6}$$

The energy of the solution to (1.1) is defind by

$$E(t,d) = (v'(t,d))^2/2 + G(v(t,d)). \tag{2.7}$$

From (2.6) and the assumption that $G(v) \geq 0$ for v sufficiently large we have

$$\lim_{n\to\infty} E(t_n,d) = \infty. \tag{2.8}$$

On the other hand, from (2.7) we obtain

$$\begin{aligned} E'(t,d) =& v'(t,d)p(t) - nt^{-1}(v'(t,d))^2 \\ &\leq \mid v'(t,d) \mid \|p\|_\infty \leq \sqrt{2}\|p\|_\infty \sqrt{E(t,d)}. \end{aligned}$$

Thus, for every $t \in [0,T]$

$$E(t,d) \leq (\sqrt{E(0,d)} + (\sqrt{2}/2)\|p\|_\infty t)^2 \leq (\sqrt{E(0,d)} + \|p\|_\infty T)^2, \tag{2.9}$$

which contradicts (2.8). Hence, there is no blow up. This fact together with Proposition 2.2. proves the theorem.

Remark 2.5. We observe that the assumption that there exists a real number $b > 0$ such that $G(v) \geq 0$ for all $v \in R$ and $\mid v \mid > b$ holds if $g(v)/v \geq M$, for v sufficiently large and $M > 0$. Moreover, it holds if g is superlinear, i. e. $\lim_{|v|\to\infty} g(v)/v = \infty$, or if g is superlinear with jumping, i. e. $\lim_{v\to-\infty} g(v)/v = M > 0$, and $\lim_{v\to\infty} g(v)/v = \infty$.

3. Applications. Let $\Omega \subset R^N$ be a bounded domain. Point in R^N are denoted by $x = (x_1, \dots, x_N)$, and $\|x\| = (x_1^2 + \cdots + X_N^2)^{(\frac{1}{2})}$. A function $u = u(x)$ defined on Ω is said to be radially symmetric if $\|x\| = \|y\|$ implies that

$$u(x_1, \dots, x_N) = u(y_1, \dots, y_N). \tag{3.1}$$

Hence

$$v(r) = u(x_1, \dots, x_N), \qquad r = (x_1^2 + \cdots + x_N^2)^{(\frac{1}{2})} \tag{3.2}$$

are well defined. Using the notation $v' = \partial v/\partial r$ and $n = N - 1$, a simple calculation shows that

$$\Delta u = \Delta v = v'' + nt^{-1}v' \tag{3.3}$$

Thus, the existence of radially symmetric solutions to a nonlinear Dirichlet problem

$$\begin{aligned} \Delta u + g(u) &= p(\|x\|) \qquad x \in \Omega \\ u &= 0 \qquad\quad x \in \partial\Omega, \end{aligned} \tag{3.4}$$

where Ω is the ball in R^N of radius T, is equivalent to solving this o. d. e. problem

$$\begin{aligned} v'' + nr^{-1}v' + g(v) &= p(r) \qquad r \in [0, T] \\ v'(0), &= 0, \qquad v(T) = 0 \end{aligned} \tag{3.5}$$

If g is superlinear or superlinear with jumping (see Remark 2.5), then the singular problem (3.5) can be solved by applying Theorem 2.1. for the initial value problem (1.1), and combining this result with the shooting method (see (1) and (2)).

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