PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série tome 45 (59), 1989, pp. 81-84

A PROPERTY OF THE CLASS OF FUNCTIONS WHOSE DERIVATIVE HAS A POSITIVE REAL PART

Milutin Obradović

Abstract. We give a subordination relation for the functions $f(z)/z$ where f belongs to the class of analytic functions in $|z|<1$ for which $Re\{f'(z)\}>0$. Some consequences are also given.

Let A denote the class of functions of the form $f(z)=z+\sum_{k=2}^{\infty}a_kz^k$ which are analytic in the unit disc $U=\{z:|z|<1\}$.

Let $f(z)$ and $F(z)$ be analytic in the unit disc U. The function $f(z)$ is subordinate to $F(z)$, if $F(z)$ is univalent, $f(0)=F(0)$ and $f(U)\subset F(U)$. For this relation the following symbol $f(z)\prec F(z)$ or $f\prec F$ is used.

For a function $f(z)\in A$ we say that it belongs to the class $P'[a,b]$, $-1\leq b<a\leq 1$ if and only if

$$f'(z)\prec(1+az)/(1+bz). \tag{1}$$

Geometrically, this means that the image of U under $f'(z)$ is inside the open disc centered on the real axis whose diametar has end points $(1-a)/(1-b)$ and $(1+a)/(1+b)$. From this we conclude that $f'(z)$ has a positive real part and it is univalent in U ([**8**]). For example, $P'[1,-1]$ is the class of functions for which $Re\{f'(z)\}>0$, $z\in U$, and $P'[1-2\alpha,-1]$ is the class for which $Re\{f'(z)\}>\alpha$, $0\leq\alpha<1$, $z\in U$. Various results for such functions are given, for example, in [**1, 3, 7**].

Further we cite the following definition [**8**]. We suppose that $f(z)$ is analytic in U. The function $f(z)$, with $f'(0)\neq 0$, is convex if and only if $Re\{1+zf''(z)/f'(z)\}>0$, $z\in U$. Such a function belongs to the class of univalent functions in U.

The first-order differential subordination with many interesting applications in considered by Miller and Mocany in [**5**]. (For the general theory of differential

AMS Subject Classification (1980): Primary 30 C 45

subordinations see [**4**]). Namely, if $\psi : \mathbf{C}^2 \to \mathbf{C}$ is analytic in a domain D, if $h(z)$ is univalent in U, and if $p(z)$ is analytic in U with $(p(z),\ zp'(z)) \in D,\ z \in U$, then $p(z)$ is said to satisfy the first order differential subordination if

$$\psi(p(z),\ zp'(z)) \prec h(z). \tag{2}$$

The univalent function $q(z)$ is said to be a dominant of the differential subordination (2) if $p(z) \prec q(z)$ for all $p(z)$ satisfying (2). If $\tilde{q}(z)$ is a dominant of (2) and $\tilde{q}(z) \prec q(z)$ for all dominants $q(z)$ of (2), then $\tilde{q}(z)$ is said to be the best dominant of (2).

In this paper we give a subordination relation for $f(z)/z$, where $f(z) \in P'[a,b]$. Also we give some estimates of $Re\{f(z)/z\}$ for certain $f(z) \in P'[a,b]$.

For these results we need the following lemma which is derived from a theorem due to Miller and Mocanu [**5,** Th 3, p. 190].

LEMMA 1. *Let $q(z)$ be a convex function in U and let $p(z)$ be analytic in U with $p(0) = q(0)$. If*

$$p(z) + zp'(z) \prec q(z) + zq'(z) \tag{3}$$

then $p(z) \prec q(z)$, and $q(z)$ is the best dominant in (3).

By using this lemma we derive

THEOREM 1. *Let $f(z) \in P'[a,b]$, $-1 \le b < a \le 1$. Then*

(i) $$\frac{f(z)}{z} \prec \frac{a}{b} + \left(1 - \frac{a}{b}\right)\frac{ln(1+bz)}{bz} \quad \textit{for } b \neq 0;$$

(ii) $$\frac{f(z)}{z} \prec 1 + \frac{a}{2}z \quad \textit{for } b = 0,$$

and these are the best dominants.

Proof. We denote by

$$q(z) = \frac{a}{b} + \left(1 - \frac{a}{b}\right)\frac{\ln(1+bz)}{bz} \quad (b \neq 0), \tag{4}$$

and we show that $q(z)$ is a convex function in U. Indeed consider the function

$$q_1(z) = \frac{2(z - \ln(1+z))}{z} = 2\left(1 - \frac{\ln(1+z)}{z}\right) = z + 2\sum_{n=2}^{\infty}(-1)^{n-1}\frac{z^n}{n+1}. \tag{5}$$

Since the function $q(z) = z/(1+z)$ is convex in U, then the function

$$G(z) = \frac{2}{z}\int_0^z q(z)dz = q_1(z)$$

is also convex in U (Libera [**2**]). Therefore, from (5) we get that $z^{-1}\ln(1+z) = 1 - q_1(z)/2$ is convex in U, and this is true for $(bz)^{-1}\ln(1+bz)$. (see [**8**], i. e. for the function $q(z)$.

Further we have that

$$q(z) + zq'(z) = (zq)' = (1 + az)/(1 + bz). \tag{6}$$

Now, let $p(z)$ be analytic in U with $p(0) = q(0) = 1$. Then from Lemma 1 we have that the following implication

$$p(z) + zp'(z) \prec (1 + az)/(1 + bz) \Rightarrow p(z) \prec q(z) \tag{7}$$

is true and that $q(z)$ is the best dominant. If we set $p(z) = f(z)/z$, then from (7) we obtain the result (i) of Theorem 1.

The proof in the case (ii) is similar as in the case (i).

If we put $a = 1 - 2\alpha,\ 0 \le \alpha < 1$ and $b = -1$, then we have the following.

COROLLARY 1. *Let $f(z) \in A$ and let $Re\{f'(z)\} > \alpha,\ 0 \le \alpha < 1$. Then*

$$f(z)/z \prec 2\alpha - 1 - 2(1 - \alpha)z^{-1}\ln(1 - z) \tag{8}$$

and this is the best dominant.

For the next corollaries of Theorem 1 we need the following

LEMMA 2. *Let $| z |< r,\ 0 < r \le 1$. Then*

$$Re\{z^{-1}\ln(1 + z)\} > r^{-1}\ln(1 + r). \tag{9}$$

This estimate is sharp.

Proof. As we showed in the proof of Theorem 1, the function $g(z) = [\ln(1 + z)]/z$ is convex in $| z |< r,\ 0 < r \le 1$. Since it has real coefficients, then the image of $| z |< r$ by $g(z)$ is convex and symmetrical with respect to the real axis. Then we have

$$\inf_{|z|<r} Re\{g(z)\} = \min\{g(-r)\} = r^{-1}\ln(1 + r),$$

and from this the estimate (9) follows.

From Theorem 1 and Lemma 2, we get directly

COROLLARY 2. *Let $f(z) \in P'[a, b]$, with $-1 \le b < a \le 1$ and $b < 0$, then*

$$Re\left\{\frac{f(z)}{z}\right\} > \frac{a}{b} - \left(1 - \frac{a}{b}\right)\frac{\ln(1 - b)}{b}, \tag{10}$$

The function $q(z)$ defined by (4) shows that the bound (10) is sharp.

Especially, for $a = 1 - 2\alpha$ and $b = -1$ from (10) of Corollary 2, we obtain

COROLLARY 3. *Let $f(z) \in A$ and $Re\{f'(z)\} > \alpha,\ 0 \le \alpha < 1$. Then*

$$Re\{f(z)/z\} > 2\alpha - 1 + 2(1 - \alpha)\ln 2,$$

and this result is sharp.

This improves an earlier result by the author [**6**].

The author would like to thank the referee for a careful reading and same useful suggestions.

REFERENCES

[1] O. M. Juneja and M. L. Mogra, *A class of univalent functions*, Bull. Sci. Math. 2e série, **103** (1979), 435–447.

[2] R. J. Libera, *Some classes of regular univalent functions*, Proc. Amer. Math. Soc. **16** (1965), 755-758.

[3] T. H. MacGregor, *Functions whose derivative has a positive real part*, Trans. Amer. Math. Soc. **104** (1962), 532–537.

[4] S. S. Miller and P. T. Mocanu, *Differential subordinations and univalent functions*, Michigan Math. J. **28** (1981), 157–171.

[5] S. S. Miller and P. T. Mocanu,*On some classes of first-order differential subordinations*, Michigan Math. J. **32** (1985), 185–195.

[6] M. Obradović, *On certain inequalities for some regular function in* $|z|<1$, Internat J. Math. Sci. **8** (1985), 677–681.

[7] K. S. Padmanabhan, *On a certain class of functions whose derivatives have a positive real part in the unit disc.* Ann. Polon. Math. **23** (1970-1971), 73–81.

[8] Ch. Pommerenke, *Univalent Functions*, Vanderhoeck & Ruprecht, Götingen, 1975.

Katedra za matematiku
Tehnološko-metalurški fakultet
11000 Beograd
Jugoslavija

(Received 28 09 1987)
(Revised 05 07 1988)