PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série tome 45 (59), 1989, pp. 65-72

ON REGULAR RINGS AND SELF-INJECTIVE RINGS, IV

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Abstract. This paper is essentially concerned with f-injectivity (a generalization of injectivity) and an analog ous concept which generalizes projectivity, called F-projectivity.

Introduction. Throughout, A represents an associative ring with identy and A-modules are unital. J and Z denote respectively the Jacobson radical and the left singular ideal of A. ${}_AM$ is called f-injective (resp. p-injective) if, for any finitely generated (resp. principal) left ideal I of A, every left A-homomorphism of I into M extends to A. Then A is von Neumann regular iff every left A-module is flat iff every left A-module is f-injective (p-injective). Right f-injective (resp. p-injective) modules are similarly defined. Flatness and f-injectivity are distinct concepts. However, if I is a left ideal of A, then ${}_AI$ f-injective implies ${}_AA/I$ flat. A is called left (resp. right) f-injective if ${}_AA$ (resp. A_A) is f-injective.

Injective and projective modules, extensively studied in recent years, are fundamental concepts in ring theory (cf. [**2, 3, 4, 8**]). Our first result gives a condition for a finitely generated left ideal of a semi-prime left f-injective ring to be generated by a central idempotent. Self-injective regular rings, biregular rings are next considered. Quasi-Frobeniusean rings are characterised in terms of F-projectivity and f-injectivity. If A has a classical left quotient ring Q such that every divisible torsionfree quasi-injective left A-module is an F-projective left Q-module, then Q is semi-simple Artinian. A sufficient condition is given for a classical quotient ring to be Noetherian. Left duo rings whose divisible left modules are f-injective are characterized.

As usual, an ideal of A means a two-sided ideal and A is called left duo (after E. H. Feller) if every left ideal of A is an ideal. A left (right) ideal is called reduced if it contains no non–zero nilpotent element. The concepts of f-injec-tivity and p-injectivity have been studied by various authors (cf. for example, [**1, 11, 12, 13, 14, 15**] and in connection with semigroup and torsion theories, consult [**5, 6, 10, 16**]. In semigroup theory, f-injectivity (p-injectivity) is also called weak

AMS Subject Classification (1980): Primary 16 A 30, 16 A 52, 16 A 36, 16 A 50

f-injectivity (weak p-injectivity) (cf. [**5**,**6**]. Note that if A is left f-injective, then $J = Z$ (cf. [**17**]).

The rings considered in the following proposition need not be von Neumann regular (cf. for example, [**11**]).

PROPOSITION 1. *Let A be a semi-prime left f-injective ring. The following conditions are then equivalent for a finitely generated left ideal F:*

(1) *F is generated by a central idempotent;*

(2) *$F = l(T)$, where T is a finitely generated left ideal.*

Proof. If $F = Ae$ where e ia a central idempotent, then $F = l(u)$ where $u = l - e$ is central and hence $F = l(AuA)$, where $AuA = Au$. This shows that (1) implies (2).

Assume (2). Then $F = l(T)$, where T is an ideal of A, ${}_AT$ is finitely generated. Therefore F is an ideal of A. By [**7**, Theorem 1], $r(F \cap T) = r(F) + r(T)$ and since A is semi-prime, $A = r(o) = r(l(T) \cap T) = r(F \cap T) = r(F) + r(T)$. Now $F = l(T) = r(T)$ and therefore $A = F + r(F)$. Since $F \cap r(F) = F \cap l(F) = 0$, then $A = F \oplus r(F)$. It follows from the semi-primeness of A that F is generated by a central idempotent.

COROLLARY 1.1. *The following conditions are equivalent for a left-f-injective ring A:*

(1) *A is biregular;*

(2) *A is semi-prime such that for each $a \in A, AaA$ is the principal left annihilator of AbA for some $b \in A$.*

The proof of Proposition 1 yields.

PROPOSITION 2. *Let A be a semi prime left self-injective ring. If T is an ideal of A which is an annihilator, then T is generated by a central idempotent. Consequently, any finitely generated right ideal which is an ideal of A is generated by a central idempotent.*

Combining the results above with [**19**, Theorem (DL)], we get

PROPOSITION 3. *If A is left self-injective, the following conditions are equivalent:*

(1) *A is regular and biregular;*

(2) *A is semi-prime such that for any $a \in A$, AaA is the principal left annihilator of AbA for some $b \in A$;*

(3) *A is semi-prime such that for any $a \in A$, AaA is a principal right ideal of A;*

(4) *If $a, b \in A$ such that $AaA + AbA \neq A$, then $AaA + AbA$ is the left annihilator of a non-nilpotent central element of A.*

Proposition 3 and Corollary 1.1. yield

Proposition 4. *The following conditions are equivalent for a left self-injective ring A:*

(1) *A is regular such that every ideal is generated by a central idempotent;*

(2) *A is right non-singular such that every ideal is a right annihilator ideal;*

(3) *A is semi-prime such that every ideal is a finitely generated right ideal of A.*

Question. If A is semi-prime, T an ideal of A which is a left annihilator, is T a complement left ideal of A? (In case A is prime, the answer is positive.)

As an analog e of f-injectivity, we introduce F-projectivity.

Definition. A left A-module P is called F-projective if, given any finitely generated left A-modules M, N with an epimorphism $g : M \to N$ and any left A-homomorphism $f : P \to N$, there exists a left A-homomorphism $h : P \to M$ such that $gh = f$.

It may be noted that a direct sum of left A-modules is F-projective if, and only if, each direct summand is F-projective.

Proposition 5. *If P is a finitely generated F-projective left A-module, then ${}_AP$ is projective.*

Proof. Let $P = \sum_{i=1}^{n} Au_i$ be a F-projective left A-module. If M, N are left A-modules, $p\ :{}_AM \to{}_AN$ an epimorphism, $f\ : P \to N$ a left A-homomorphism, set $f(u_i) = v_i$ for each i, $1 \leq i \leq n$. Since p is an epimorphism, there exists $w_i \in M$ such that $p(w_i) = v_i$, $1 \leq i \leq n$. If $M' = \sum_{i=1}^{n} Aw_i$, $N' = \sum_{i=1}^{n} Av_i$, then p', the restriction of p to M', is an epimorphism of M' onto N'. Since ${}_AP$ is F-projective, there exists a left A-homomorphism $h' : P \to M'$ such that $p'h' = f$. If j is the inclusion map $M' \to M$, $h = jh'$, for any $y \in P$, $y = \sum_{i=1}^{n} a_iu + i$, $ph(y) = p(\sum_{i=1}^{n} a_ih(u_i)) = p(\sum_{i=1}^{n} a_ih'(u_i)) = p'(\sum_{i=1}^{n} a_ih'(u_i)) = \sum_{i=1}^{n} a_ip'h'(u_i) = \sum_{i=1}^{n} a_if(u_i) = f(y)$, which proves that $ph = f$. The proposition then follows.

Corollary 5.1. *If A is commutative, then A is quasi-Frobeniusean iff A is Artinian such that every injective A-module if F-projective.*

Proof. Assume that A is Artinian and every injective A-module is F-projective. If M is injective, then M is a direct sum of finitely generated A-modules M_i. Since M is F-projective, then each M_i is F-projective which implies that M_i is projective, whence M is a projective A-module. The corollary then follows from [**3**, Theorem 24.20].

If A is left perfect, left f-injective, then every simple left A-module is a homomorphic image of an injective left A-module (cf. [**2,** P. 481]). If $\times$ is right f-injective satisfying the maximum condition on left annihilator ideals, then A is quasi-Frobeniusean [**18**]. Recall that (1) A is a left Kasch ring if every maximal left ideal of A is a left annihilator; (2) A is right pseudo-coherent iff the right annihilator of every finitely generated left ideal is a finitely generated right ideal; (3) A left

A-module C is a cogenerator if, for any M in the category of left A-modules, there exists a monomorphism of M into a direct product of copies of C. A is left pseudo-Frobeniusean iff ${}_AA$ is an injective cogenerator iff A is a left self-injective left Kasch ring. Quasi-Frobeniusean rings are both left and right pseudo-Frobeniusean.

A is called left (resp. right) hypercyclic if the injective hull of every cyclic left (resp. right) A-module is cyclic.

Remark 1. The following conditions are equivalent: (1) Every factor ring of A is quasi-Frobeniusean; (2) A is a left and right hypercyclic principal left ideal ring such that every injective one-sided A-module is F-projective. (cf. [**3**, Proposition 25.4. 6 B]).

If A is von Neumann regular, a theorem of I. Kaplansky asserts that for any projective left A-module P, every finitely generated left submodule is a direct summand. Consequently, the following remark holds.

Remark 2. If A is von Neumann regular, then every F-projecrtive left A-module is projective.

For results on coherent rings, consult [**11**].

THEOREM 6. *The following conditions are equivalent:*

(1) *A is quasi-Frobeniusean;*

(2) *A is coherent left f-injective left cogenerating such that every flat left A-module is F-projective;*

(3) *A is left cogenerating such that every injective left A-module is F-projective;*

(4) *A is right f-injective left perfect right pseudo-coherent;*

(5) *A is left f-injective left perfect right pseudo-coherent.*

Proof. Obviously, (1) implies (2).

Assume (2). Since A is left cogenerating, then every left ideal is a left annihilator. Since A is left f-injective, then every finitely generated right ideal is a right annihilator. By [**11**, Theorem 2.1], every injective left A-module is flat. Therefore (2) implies (3).

Assume (3). Since ${}_AA$ is a cogenerator, for any non-zero left A-module M, any $o \neq y \in M$, there exist a left A-homomorphism $g : M \to A$ such that $g(y) \neq o$. Now if ${}_AP$ is F-projective, for any left A-module M with an epimorphism $p : M \to P$, since there exists a non-zero left A-homomorphism $g : M \to A$, we can show that p splits. But then, we get ${}_AP$ projective. Consequently, every injective left A-module is projective and by [**3**, Theorem 24.20], A is quasi-Frobeniusean. Therefore (3) imlies (4).

Assume (4). Let $I_1 \subseteq I_2 \subseteq \ldots\ldots \subseteq I_n \subseteq \ldots\ldots$ be an ascending chain of finitely generated left ideals of A. If $R_i = r(I_i)$ for each i, since A is right pseudo-coherent, then R_i is a finitely generated right ideal for each i. Since A is left perfect, then $R_1 \supseteq R_2 \supseteq R_3 \ldots\ldots \supseteq R_n \supseteq \ldots$ yields $R_m = R_s$ for some positive integer m and all $s \geq m$ [**8**, P. 303]. Now A right f-injective implies that every finitely generated left ideal is a left annihilator, whence $I_m = 1(r(I_m)) = 1(R_m) = 1(R_s) = 1(r(I_s)) = I_s$ for all $s \geq m$. This proves that A is left Noetherian. Then (4) imlies (5) by [**18**, Theorem 2].

Assume (5). Let R be a right annihilator ideal, U a subset of A such that $R = r(U)$. Since A is right pseudo-coherent, the right annihilator for any finitely generated left ideal is finitely generated. Since A is left perfect, there exists a finite subset F of U such that $R = r(F)$ which implies that R must be a finitely generated right ideal. Therefore A satisfies the minimum condition on right annihilators which implies that A satisfies the maximum condition on left annihilators. Then Z is nilpotent and since A is left f-injective, $Z = J$ [**17**, Proposition 3] which implies that A is semi-primary. Therefore A is a left f-injective semi-primary ring which implies that A is left Kash. Now let F be a finitely generated left ideal of A. Suppose that $F \subset 1(r(F))$. If $y \in 1(r(F)), y \notin F$, $G = F + Ay$, the set E of left ideals I of A such that $F \subseteq I \subset G$ is an inductive set and by Zorn Lemma, E has a maximal member K. Then G/K is a simple left A-module and since A is left Kash, $G/K = Au$, where Au is a minimal left ideal of A. There exists a left A-homomorphism $G \to Au$ which yields a non-zero left A-homomorphism $f : G \to A$ such that $f(K) = o$. Since G is a finitely generated left ideal of A, there exists $c \in A$ such that $f(b) = bc$ for all $b \in G$ (A being left f-injective). Then $f(F) = o$ which implies $c \in r(F)$, whence $f(y) = yc = o$. Thus $f(G) = o$ which contradicts f non-zero. This proves that F must be a left annihilator. Then A satisfies the ascending chain condition on finitely generated left ideals which implies that A is left Noetherian. Therefore A is left self-injective, left Noetherian and hence (5) implies (1).

We now consider classical quotient rings in terms of F-projectivity and f-injectivity. For the definition and properties of classical quotient rings, consult, for example, [**4**] and [**9**]. For divisibility and torsionfreeness, see [**9**].

Proposition 7. *Let A have a classical left quotient ring Q. The following conditions are then equivalent:*

(1) *Q is semi-simple Artinian;*

(2) *Every divisible torsionfree quasi-injective left A-module is an F-projective left Q-module.*

Proof. Since any divisible torsionfree left A-module is left Q-module [**9**, p. 140], (1) implies (2).

Assume (2). Let P be a quasi-injective left Q-module. Since any left Q-module is a divisible torsionfree left A-module, then ${}_AP$ is divisible torsionfree. If ${}_AN$ is a submodule of ${}_AP, f : N \to P$ a left A-homomorphism, since ${}_AN$ is torsionfree, we may define a left Q-homomorphism $g : QN \to P$ by $g(c^{-1}w) = c^{-1}f(w)$, for any non-zero divisor c of A, $w \in N$. Since ${}_QP$ is quasi-injective, g extends to an endomorphism of ${}_QP$ and so f extends to an endomorphism of ${}_AP$. This shows that ${}_AP$ is quasi-injective and by hypothesis, ${}_QP$ is F-projective.

Therefore every simple left Q-module, being quasi-injective, is F-projective and by Proposition 5, every simple left Q-module is projective. This proves that (2) implies (1).

PROPOSITION 8. *Let A have a classical left quotient ring Q. If every divisible torsionfree f-injective left A-module is injective, then Q is left Noetherian.*

Proof. Let M be an f-injective left Q-module, $F = \sum_{i=1}^{n} Au_i$ a finitely generated left ideal of A, $g :_A F \to_A M$ a left *A-homomorphism.* Define $G : \sum_{i=1}^{n} Qu_i \to M$ by $G(\sum_{i=1}^{n} q_i u_i) = \sum_{i=1}^{n} q_i(g(u_i))$ $q_i \in Q,\ 1 \leq i \leq n$. Then G is a well-defined left Q-homomorphism and so there exists $y \in M$ such that $G(u_i) = u_i y$ for each $i,\ 1 \leq i \leq n$. Consequently, for any $v \in F$, $v = \sum_{i=1}^{n} a_i u_i$, $a_i \in A$, $g(v) = \sum_{i=1}^{n} a_i g(u_i) = \sum_{i=1}^{n} a_i G(u_i) = \sum_{i=1}^{n} a_i u_i y = (\sum_{i=1}^{n} a_i u_i)\ y = vy$ which proves that ${}_AM$ is *f-injective.* Since ${}_AM$ is divisible torsionfree, then ${}_AM$ is injective. If $j :_Q P \to_Q N$ is a monomorphism, $f :_Q P \to_Q M$ a left Q-homomorphism, $\hat{j}, \hat{f}$ the restrictions of j, f respectively to ${}_AP$, since ${}_AM$ is injective; there exists a left A-homomorphism $\hat{h} :_A N \to_A M$ such that $\hat{h}\hat{j} = \hat{f}$. If $h :_Q N \to_Q M$ is defined by $h(qu) = gh(u)$ for all $q \in Q$, $u \in N$, then $hj = f$ which proves that ${}_QM$ is injective. Since any direct sum of f-injective left Q-modules is f-injective, then any direct sum of injective left Q-modules is injective which implies that Q is left Noetherian [**3**, Theorem 20.1].

We now turn to rings whose divisible modules are f-injective.

PROPOSITION 9. *If every divisible left A-module is f-injective, then A is left semi- hereditary.*

Proof. Since any injective left A-module is divisible and any quotient module of a divisible module is divisible, then the quotient of any injective left A-module is f-injective. Let I be a finitely generated left ideal of A. Given any quotient module M/N of a left A-module $M, k : M \to M/N$ the natural projection, if $f : I \to M/N$ is a left A-homomorphism, E the injective hull of ${}_AM, j : M/N \to E/N$ the inclusion map, then with $F = jf$, we have $F : I \to E/N$. Therefore ${}_AE/N$ is f-injective which implies the existence of $\bar{z} = z + N$, $z \in E$, such that $F(b) = b\bar{z}$ for all $b \in I$. Define $g : I \to E$ by $g(b) = bz$ for all $b \in I$. If $K : E \to E/N$ is the natural projection, for all $b \in I$, $Kg(b) = K(bz) = bK(z) = b\bar{z} = F(b)$, whence $F = Kg$. As $f(I) \subseteq M/N$, then $F(I) \subseteq M/N$, $g(I) \subseteq M$ and if define $h : I \to M$ by $h(b) = g(b)$ for all $b \in I$, we get $kh = f$. This proves that ${}_AI$ is projective which yields the proposition.

The next result gives a characterization of commutative rings Mhocse divisible modules are f-injective. It is clear that over von Neumann regular rings, p-injectivity coincides with f-injectivity.

THEOREM 10. *The following conditions are equivalent for a ring A whose complement left ideals are ideals:*

(1) *Every divisible left A-module is f-injective;*

(2) *A is left semi-hereditary and every p-injective left A- module is f-injective;*

(3) *Every p-injective left A-module is f-injective and for every $a \in A$, there exist an idempotent $e \in 1(a)$ and n elements $b_1, \ldots\ldots, b_n$ in Aa, n non zero-divisors $c_1, \ldots\ldots, c_n$ in A such that $a = \sum_{i=1}^n a_i b_i, a_i \in A, (1-e)a_i c_i = a$ for each i, $1 \leq i \leq n$.*

Proof. Since any p-injective left A-module is divisible, then (1) implies (2) by Proposition 9.

Assume (2). Let D be a divisible left A-module. For any $a \in A$, since ${}_AAa$ is projective, $1(a) = Ae$, $e = e^2 \in A$. Since A is left non-singular and every complement left ideal is an ideal of A, then A is reduced which implies that e is a central idempotent. With $c = a + e$, since $Aa \cap 1(a) = o$, it is easy to see that c is a non zero-divisor of A. If $f : Aa \to D$ is a left A-homomorphism, set $f(a) = d \in D$. Then $d = cv$ for some $v \in D$ (inasumuch as ${}_AD$ is divisible) and $a = (1-e)$ a implies $f(a) = f((1-e)a) = (1-e)f(a) = (1-e)cv = (1-e)\,(a+e)v = av$ which shows that ${}_AD$ id p-injective. By hypothesis, ${}_AD$ is f-injective. Thus (2) implies (3) (it is sufficient to take $n = 1,\ b_1 = a,\ c_1 = c,\ a_1 = 1$).

Assume (3). Let M be a divisible left A-module, $a \in A, f :\ Aa \to M$ a left A-homomorphism. By hypothesis, there exist a positive integre $n, b_1, \ldots\ldots, b_n \in Aa$, an idempotent e such that $ea = o$, nono zero-divisors $c_1, \ldots\ldots, c_n \in A$ such that $a = \sum_{i=1}^n a_i b_i,\ a_i \in A, (1-e)a_i c_i = a$ for each i, $1 \leq i \leq n$. Since ${}_AM$ is divisible, for each i, $1 \leq i \leq n, f(b_i) = c_i m_i, m_i \in M$ and $f(a) = f((1-e)a) = (1-e)f(a) = (1-e)f(\sum_{i=1}^n a_i b_i) = (1-e)(\sum_{i=1}^n a_i f(b_i)) = (1-e)(\sum_{i=1}^n a_i c_i m_i) = a(\sum_{i=1}^n m_i)$ which proves that ${}_AM$ is p-injective. By hypothesis, every divisible left A-module is f-injective and hence (3) implies (1).

COROLLARY 10.1. *If every complement left ideal of A is an ideal, the following are then equivalent: (1) Every divisible left A-module is injective; ((2) A is a left hereditary, left Noetherian ring whose p-injective left modules are f-injective.*

COROLLARY 10.2. *A left duo ring whose divisible left modules are injective admits a classical left quotient ring which is a finite direct sum of division rings.*

The next remark completes nicely Theorem 4 of [**20**].

Remark 3. The following conditions are equivalent: (a) A is left self-injective regular with non-zero socle; (b) A is left f-injective with an injective non-singular maximal left ideal; (c) A has an injective non-singular maximal left ideal and the left socle of A is f-injective; (d) A is semi-prime with an injective non-singular, maximal left ideal such that every maximal right ideal is f-injective.

Remark 4. A is strongly regular iff A is a left f-injective ring with a nont-singular maximal left ideal such that every maximal left ideal is an ideal of A.

Remark 5. Let A be left f-injective. If every finitely generated faithful left A-module is F-projective, then A is left self-injective and consequently, every finitely generated faithful left A-module is injective.

Remark 6. (1) A is von Neumann regular iff every divisible left A-module is flat; (2) (he following conditions are equivalent: (a) A is semi-simple Artnian; (b) every divisible left A-module is projective; (c) every left A-module is F-proective.

REFERENCES

[1] G. Baccela, *Generalized V-rings and von Neumann regular rings*, Rend. Sem. Mat. Univ. Padova **72** (1984), 117–133.

[2] H. Bass, *Finitistic dimension and a homological generalization of semi-primary rings*, Trans. Amer. Math. Soc. **95** (1960), 466–488.

[3] C. Faith, *Algebra II: Ring Theory*, Springer-Verlag, **191** (1976).

[4] K. R. Goddearl, *Ring Theory, Nonsingular rings and modules*, Pure Appl. Math. **33** (1976).

[5] V. Gould, *The characteristation of monoids by properties of their S-systems*, Semigroup Forum **32** (1985), 251–265.

[6] V. Gould, *Divisible S-systems and R-modules*, Proc. Edinburg Math. Soc. **30** (1987), 187–200.

[7] M. Ikeda and T. Nakayama, *On some characteristic properties of quasi-Frobenius and regular rings*, Proc. Amer. Math. Soc. **5** (1954), 15–19.

[8] F. Kasch, *Modules and Rings* London Math. Soc., 1982.

[9] L. Levy, *Torsion free and divisible modules over non-integral domains*, Canad. J. Math. **15** (1963), 132–151.

[10] J. K. Luedeman, F. R. McMorris, and S. K. Sim, *Semi groups for which every totally irreductible S-system is injective*, Comment. Math. Univ. Carolinea **19** (1978), 27–35.

[11] J. L. G. Pardo and N. R. Gonzales, *On some properties of IF rings*, Quaestiones Math. **5** (1983), 395–405.

[12] Sii-Joo Kim, *A note on generalizations of PCI rings*, Honam Math. **5** (1983), 205–214.

[13] A. K. Tiwary, S. A. Paramhans and B. M. Pandey, *Generalizations of quasi-injectivity*, Progress Math. **13** (1979), 31– 40.

[14] H. Tominaga, *On s-unital rings*, Maths. J. Okayama Univ. **18** (1976), 117-134.

[15] H. Tominaga, *On s-unital rings, II*, Math. J. Okayama Univ. **19** (1977), 171–182.

[16] K. Varadarajan and K. Wehrhahn, *P-injectivity of simple pre-torsion modules*, Glasgov Math. J. **28** (1986), 223–225.

[17] R. Yue Chi Ming, *On von Neumann regular rings, VIII*, J. Korean Math. Soc. **19** (1983), 97–104.

[18] R. Yue Chi Ming, *On quasi-Frobeniusean and Artinian rings*, Publ. Inst. Math. (Beograd) **33 (47)** (1983), 239–245.

[19] R. Yue Chi Ming, *On regular rings and self-injective rings, II*, Glasnik Mat. **18 (38)** (1983), 221–229.

[20] R. Yue Chi Ming, *On regular rings and self-injective rings, III*, Tamkang J. Math. **17** (1986), $n^o 3$, 59–67.

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(Received 07 09 1987)