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ON CATEGORIES OF RELATIONS

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Abstract. This paper is concerned with relations in categories with pullbacks, studied by Kawahara in [**3**], and with a kind of congruences that may be considered in the corresponding category of spans. Also, having mind G. Conte's results about symmetrizations of categories [**2**], some categories of relations are compared.

0. Introduction. The classical example of a relation between two sets A and B is defined as a monosubobject [**7**] of the cartesian product of $A \times B$. In that case, relations are composable by pulbacks, and they form an involutive category in which the category $\mathcal{S}$ of sets and functions may be embedded. For a category K with pullbacks, relations may be defined by pairs of K-morphisms $A \leftarrow X \rightarrow B$ under a suitable equivalence relation. When composition of equivalence classes by pullbacks is defined, the corresponding category of relations form an involutive category in which K is embeddable [**3**].

Since any pair of K-arrows $A \leftarrow \cdot \rightarrow B$ may be considered as a functor-object of the functor-category $K^{\leftarrow \cdot \rightarrow}$, it is natural and useful to define an equivalence relation by the suitable natural transformations. That kind of equivalence relation and the corresponding Kawahara quotient category of relations are considered in this paper. The graph functor G of the embedding of the category K into the category of relations of K is universal among the relational functors from K to some involutive cattegory V. That fact, using similar results by Kawahara [**3**] is proved here in a strictly categorical manner.

1. Category of spans. Let K be a category with pullbacs, V an involutive category and $F : K \rightarrow V$ a functor.

1:1. F is a relational functor if and only if it satisfies the following conditions:

$(RF1)$ For each K-morphism $f, F(f^{\$} F(f) \leq 1$, where 1 denotes the identity morphism and \$ denotes involution in V.

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$(RF2)$ If (x, y, f, g) is a pullback square $(fx = gy)$ in $K, F(f)F(g)^{\$} = F(x)^{\$}F(y)$.

A restractive subcategory E is a subcategory of a category K satisfyng the following three conditions:

$(E1)$ Iso $(K) \subset E \subset$Epi(K);

$(E2)$ if fg is an E- morphism, then g is an E-morphism;

$(E3)$ if (x, y, f, g) is a pullback in K and f is an E-morphism, then y is an E-morphism.

1:2. Let $\mathcal{R}(K)$ be a *category (of spans over* K*)* with the same objects as a functor category $K^{\leftarrow\cdot\rightarrow}$ but with composition defined by pullbacks. Namely, $(f, g) \cdot (h, k) :\, = (fp, kq)$, where (f, g) and (h, k) are objects of the category $K^{\leftarrow\cdot\rightarrow}$ (spans) and (p, q, g, h) is a pullback square over the pair of morphisms g and h with the same codomain. The set of all spans between two K- objects A and B is denoted by $\mathcal{R}(A, B)$. Composition is well defined, associativity follows from the well-known properties of pullbacks, and units are spans of the form (1, 1) where 1 is the identity on X, for any object X of K.

1:3. Let I denote a diagram category of the form $\leftarrow \cdot \rightarrow$. Since any span in $\mathcal{R}(A, B)$ is defined to be an image of the diagram category I under some functor $(f, g) :\ I \rightarrow K$ given by $(f, g) : (\leftarrow \cdot \rightarrow) \rightarrow (A \leftarrow X \rightarrow B)$, an order relation in $\mathcal{R}(A, B)$ may be defined as follows:

$(f, g) \leq (f', g')$ if and only if there exists a functor (x, y) from $\mathcal{R}(A, B)$ and a pair of natural transformations:

$$s : (x, y) \dot{\rightarrow} (f', g') \text{ and } e : (x, y) \dot{\rightarrow} (f, g)$$

where s-components are from the category K but e-components are from the retractive subcategory E. Denote that by $(s, e) : (f, g) \leq (f', g')$.

1:4. *Let* $(f, g), (f', g')$ *and* (f'', g'') *be spans from* $\mathcal{R}(A, B)$ *and let* (h, k) *be a span from* $\mathcal{R}(B, C)$*. Then:*

(i) $(f, g) \leq (f, g)$*;*

(ii) *if* $(f, g) \leq (f', g')$ *and* $(f', g') \leq (f'', g'')$ *then* $(f, g) \leq (f'', g'')$*;*

(iii) *if* $(f, g) \leq (f', g')$*, then*

(a) *if we let* t *be a* K*-morphism with* $\mathrm{cod}(t) = \mathrm{dom}(f)$ *and define* $(f, g)t$ *by* (ft, gt)*, there exists an* K*-morphism* $\hat{t}$ *such that* $(f, g)t \cdot (h, k) = ((f, g) \cdot (h, k))\hat{t}$,

(b) $(f, g) \cdot (h, k) \leq (f', g') \cdot (h, k)$.

Proof. (i) Take $(x, y) = (f, g)$ and $s = e = 1$ (with all components identities) and use the fact that identities are epimorphisms.

(ii) Let $\langle s, e\rangle : (f, g) \leq (f', g')$ and $\langle s', e'\rangle : (f', g'\rangle \leq (f'', g'')$. Constructing pullbacks (u, v, s, e') (on all components of the natural transformations s and e') one gets a new pair natural transformations $\langle s'v, eu\rangle : (f, g) \leq (f'', g'')$, where the natural transformation $s'v$ has all components from the category K, but the transformation eu has components from the retractive subcategory E (from the properties of pullbacks of the subcategory E).

(iii) (a) The functor $(f,g)t := (ft, gt) : I \to K$ is well defined and $(f,g)t$ belongs to $\mathcal{R}(A,B)$. Further, $(f,g)t \cdot (h,k) := (ft, gt) \cdot (h,k) := (fta, kb) = (f\hat{t}\hat{x}, kb) = (fx\hat{t}, kb) = (fx\hat{t}, ky\hat{t})((f,g) \cdot (h,k))\hat{t}$, where connecting pullbacks (x,y,g,h) and $(\hat{x}, \hat{t}, t, x)$, the pullback $(\hat{x}, yt, gt, h)$ is given and from their uniqueness the preceding equalites are true.

(b) Since $\langle s, e\rangle : (f,g) \le (f',g')$, by (a) there are morphisms $\hat{s}_X$ and $\hat{e}_X$ such that $(f,g)e_X \cdot (h,k) = ((f,g) \cdot (h,k))\hat{e}_X$, and $(f,g)s_X \cdot (h,k) = ((f,g) \cdot (h,k))\hat{s}_X$. The arrow e_X is an E-morphism; so by the properties of E and pullbacks, $\hat{e}_X$ is an E-morphism. By this, a new pair of natural transformations is defined $\hat{s} = (s_A, \hat{s}_X, 1_C)$ and $\hat{e} = (e_A, \hat{e}_X, 1_C)$ such that

$$\langle s, e\rangle : (f,g) \cdot (h,k) \le (f',g') \cdot (h,k).$$

1:5. Let $(f,g)^{\$} = (g,f)$. Then clearly, for (f,g) from $\mathcal{R}(A,B)$ and (f',g') from $\mathcal{R}(B,C)$, (i) $((f,g) \cdot (f',g'))^{\$} = (f',g')^{\$} \cdot (f,g)^{\$}$; (ii) $(f,g)^{\$\$} = (f,g)$ and (iii) if $(f,g) \le (f',g')$ then $(f,g)^{\$} \le (f',g')^{\$}$.
Therefore $\mathcal{R}(K)$ is an involutive category.

2. Category of relations. 2:1. Two spans (f,g) and (f',g') from $\mathcal{R}(A,B)$, are *equivalent*, i e. $(f,g) \sim (f',g')$, if and only if both $(f,g) \le (f',g')$ and $(f',g') \le (f,g)$. Note that in the case $e_A = s_A = 1_A$ and $e_B = s_B = 1_B$ one gets Kawahara's equivalence relation [**3**].

2:2. Lemma *The relation $\sim$ is an equivalence relation on $\mathcal{R}(A,B)$.*

The equivalence class of a span (f,g) is called an I-relation between A and B relative to E (abrev. "relation" from A to B) and denoted by $[f,g]$. The class of relations from A to B in K is denoted by $\mathrm{Rel}_E(A,B)$.

2:3. *Example.* As it is well-known, a multivalued function between two sets X and Y may be considered as a binary relation $r : R \to X \times Y$, where r is monic and $r_X = p_X\,|_R, r_Y = p_Y\,|_R$ (p_X and p_Y are projections). On the other hand, the only retractive subcategory in Set is Epi(Set) and therefore the equivalence class of a span (r_X, r_Y) will be a relation R from X to Y.

2:4. For two relations $[f,g]$ from $\mathrm{Rel}_E(A,B)$ and $[h,k]$ from $\mathrm{Rel}_E(B,C)$ *composition* (join operation) is defined as the following equivalence class: $[f,g] \cdot [h,k] = [fp, kq]$ where (p,q,g,h) is the pullback square defined over the pair of arrows (g,h).

2:5. Proposition. *The composition of relations is well-defined; associativity holds and* [1,1] *is the unit of this operation.*

Proof. Let $[f,g]$ be from $\mathrm{Rel}(A,B)$ and $[h,k]$ from $\mathrm{Rel}(B,C)$, and let $(f,g) \sim (f',g')$ and $(h,k) \sim (h',k')$. The construction of the pullbacks over the pairs

$(g,h),(e,p)$ and (q,e') may be followed in the diagram (2:6). Connecting these pullbacks, as shown in (2:7), and using the properties of pullbacks, we get that $e_l t'$ is an E-morphism and there exists a pair of natural transformations $d = (e_A, e_l t', e_C)$ and $r = (S_A, s_l s', s'_C)$ such that

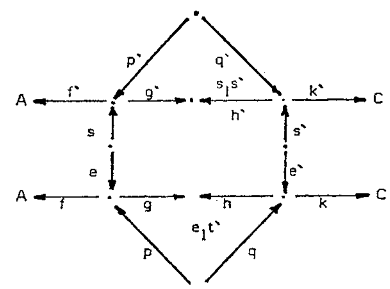
$$\langle r, d\rangle : (f,g)\cdot(h,k) \leq\leq (f',g')\cdot(h',k),$$

and similary,

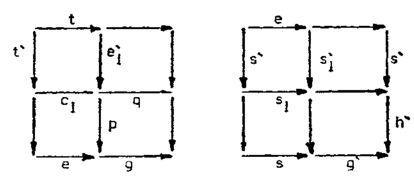
$$\langle r', d'\rangle : (f',g')\cdot(h,k) \leq (f,g)\cdot(h,k).$$

Associativity follows from the associativity of the composition of the corresponding spans. It is easy to see that $[1_A,1_A]\cdot[f,g] = [f,g]$ and $[f,g]\cdot[1_B,1_B][f,g]$ for a relation $[f,g]$ from $\mathrm{Rel}_E(A,B)$.

(2:6)



(2:7)



2:8. The preceding proposition shows that objects of $\mathcal{R}(K)/\sim$, with composition defined form a category denoted by $\mathrm{Rel}(K,E)$, and called the category of relations in K over a retractive subcategory E (Kawahara [**3**]).

2:9. PROPOSITION (Kawahara [**3**]) *There exists a contravariant rational embedding functor $G : K \to Rel(K,E)$ defined by $G : (A \to B) \to [1_A, f]$.*

2:10. PROPOSITION. *The following properties hold:*

(i) $G(1_A) = [1_A, 1_A]$, (ii) $[f,g] = G(f)^{\$}G(g)$,

(iii) $G(f)^{\$}G(f) = [1_B, 1_B]$ *if and anly if f is an E-morphism,*

(iv) *G is the relational functor,*

(v) *$G(f)$ is a retract in* $\mathrm{Rel}(K,E)$ *if and only if f is an E-morphism,*

(vi) *for each $i : A \overset{\cong}{\to} A, j : B \overset{\cong}{\to} B$, we have $G(jfi) = G(f)$.*

The proof is quite simple manipulation with given facts.

The idea for the following proposition is from Kawahara [**3**], but the statement and the proof are appropriate to our approach.

2:11. PROPOSITION. *Let K be a category with pullbacks, E a retractive subcategory of K and* Inv *the category of involutive categories and involutive functors.*

(a) *The relational functor $G : K \to$*Rel(K, E) *is the universal functor between all relational functors F from the comma category* $(K \downarrow \text{Inv})$ *with the property*

(P) *if f is an E-morphism, then $F(f)^{\$}F(f) \cong 1$.*

(b) *The free* Inv*-object over K, with respect to the forgetful functor U* :In$\to$Kat *is a pair* $(\text{Rel}(K, E), G)$.

Proof. The functor $G : K \to \text{Rel}(K, E)$ is universal between relational functor from $(K \downarrow \text{Inv})$ with the property (P) is for any relational functor $F : K \to V, V$ being an involutive category, there exists a relational functor F' :Rel$(K, E) \to V$ such that the diagram (2:12) commutes.

$$\text{Inv} \xrightarrow{\quad U \quad} \text{Kat}$$

$$(2{:}12) \qquad \begin{tikzcd} \text{Rel}(K,E) \arrow[d, "F'"] \\ V \end{tikzcd} \qquad \begin{tikzcd} K \arrow[r, "G"] \arrow[dr] & U\text{Rel}(K,E) \arrow[d, "UF'"] \\ & UV \end{tikzcd}$$

The functor F' defined by $F'[f, g] = F(f)^{\$}F(g)$ is well defined. For, if $\langle s, e\rangle$: $(f, g) \leq (f', g')$, then in the category A,

$$\begin{aligned} F'[f, g] = F(f)^{\$}F(g) = F(f)^{\$}(e_R)^{\$}F(e_R)F(g) = F(fe_R)^{\$}F(ge_R) \cong \\ \cong F(f')^{\$}F(s)^{\$}F(s)F(g') \leq F(f')^{\$}F(g') = F'[f', g']. \end{aligned}$$

Similarly, the converse relational is true, and, obviously, the definition doesn't depend of the choice of representative. Since $F'[1, 1] = 1, F'[g, f]$ and $F'(G(f)) = F(f)$, the diagram (2:12) commutes. It remains to prove that F' is an involutive functor, namely,

$$\begin{aligned} F'([f, g] \cdot [h, k]) &\cong F'(G(fp)^{\$}(kq)) \cong F(fp)^{\$}F(kq) \\ &\cong F(f)^{\$}F(p)^{\$}F(q)F(k) \cong F(f)^{\$}F(g)F(h)^{\$}F(k) \\ &\cong F'[f, g] \cdot F'[fp, kq] \end{aligned}$$

where (p, q, g, h) is the pullback square.

In terms of universal arrows, $G : K \to V$ may be universal among relational functors with a specific property in the comma category $(K \downarrow \text{Ivn})$. In that case, it is possible to define the retractive subcategory E of the category K, so that the given involutive category V is isomorphic to the category of relations Rel (K, E).

2:13. COROLLARY. *Let K be a category with pullbacks and $F : K \to A$ the relational functor which is universal among the functors in the comma category*

($K \downarrow$ Inv) *with the property: If* $F(f)^{\$}F(f) \cong 1$ *then* $H(f)^{\$}H(f) \cong 1$. *Then, there exists a retractive subcategory* E *of the category* K *and a unique isomorphism* $A \cong$Rel (K, E) *such that the diagram* (2:14) *it commutative if and only if the following holds:*

(PP) *If* $1 < eF(f)$ *for some* E*-morphism* e*, then* $F(f)^{\$}F(f) \cong 1$.

(2:14)
$$\begin{tikzcd} K \arrow[rr, "G"] \arrow[dr, "F"'] & & \mathrm{Rel}(K,E) \arrow[dl, "\mathfrak{F}"] \\ & A & \end{tikzcd}$$

The proof is a routine matter if a retractive subcategory E of the category K iz defined by $E = \{f \in K : F(f)^{\$}F(f) \cong 1\}$ as in [**3**].

2:15. *Examples*. (a) In the abelian category M (e. g. the category of all R-modules) one may define the additive relations $f : A \to B$ to be the submodules of the direct sum $A+B$ (MacLane [**6**]). Taking in M a retractive subcategory Epi(M), the category Rel(M, Epi(M)) is equal to the category of additive relation. Each $f : A \to B$ in M determines the idempotens $f^{\$}f$ and $ff^{\$}$ (where $f^{\$}$ is the converse of f) which in MacLane's model [**6**] represent the subquotients Dom$f$/Ker$f$ and Imf/Ker$f$, so that f may be called a graph if $f^{\$}f \supset 1 ff^{\$} \subset 1$. Clearly, for the relational embedding functor $G : M \to$Rel(M, Epi(M)), $G(f) = [1_A, f], G(f)^{\$}G(f) \cong 1$ if and only if $f \in$Epi(M), but for every f in $M, G(f)^{\$}G(f) \supset 1$ and $G(f)G(f)^{\$} \subset 1$.

(b) It is clear that the category of topological spaces and continuous maps, denoted by Top, has pullbacks and the subcategory of epimorphisms is a retractive subcategory of Top. So, it is possible to construct the category of relations, Rel(Top, Epi(Top)).

In the category of Hausdorff's spaces and continuous maps, denoted by H, the subcategory Epi(H) doesn't satisfy the axiom (E3) for a retractive subcategory, but, the class of all epimorphisms f such that any pullback $(\hat{x}, \hat{f}, f, x)$ implies that f is an epimorphism, is the largest retractive subcategory contained in Epi(H). Denote that subcategory by $E(H)$. The category Rel($H, E(H)$) is well defined. In both categories Top and H, the class of all regular epimorphisms is not a retractive subcategory. Since the regular epimorphisms are coequalizers for some pairs of morphisms, they are just the class of identification maps and therefore they satisfy axioms ($E1$) and ($E2$), but not ($E3$). The largest retractive subcategory E_r contained in the class of regular epimorphisms may be constructed and therefore the categories Rel(Top, E_r(TOP)) and Rel($H, E_r(H)$) of relations relative to E_r are well-defined. In both of them every relation $[f, g]$ with the properties $[f, g]\cdot[g, f] \geq 1$ and $[g, f] \cdot [f, g] \leq 1$ is a morphism t in Top(i.e. in H) such that $[f, g] = G(t)$, where G is the corresponding graph-relational functor, G :Top$\to$Rel(Toop, E_r(Top)), i. e. $G : H \to$Rel($H, E_r(H)$). If such a relation $[f, g]$ is called proper note that a proper relation is not always a continuous map (since not every epimorphism is a regular epimorphism), (see[**3**]).

3. Relations by symmetrizations of categories. In the approach by Conte [**2**] a category of relations for the given category K is viewed as the quotient, by means of a suitable equivalence relation of a bigger involution category containing K. This involution category, denoted by K^M is called the maximum symmetrization of K (it is a kind of quotient ("free") category (Brinkmann [**1**]) and any equivalence relation in K^M compatible with composition and involution is called a congruence.

3:1. A *symmetrisation* of a category K is defined as a pair $(S, \$)$ such that the following conditions are satisfied:

$(S1)$ $S : K \to A$ is a functor injective on the objects,

$(S2)$ $\$: A \to A$ is a contravariant endofunctor identical on the objects and such that $\$\$ \cong 1_A$,

$(S3)$ A is the last involutive subcategory containing $S(K)$,

$(S4)$ $\$S(u) \cong S(u^{-1}$ for any K-morphism u.

The categories of relations constructed in Kawahara [**3**] and Klein [**4**] and also in our approach may all be defined by congruences of the same kind, depending on the choice of a subcategory of the considered category K. One of the conditions is similar to the $(RF1)$ (1:1) condition for the relational functor, and the other one connects the arrows of two spans with the same end, whenever the natural transformation with E-arrows is given between them. This last one is the Ore-like condition (A) of Klein.

3:2. Let E be any subcategory of a category K. E defines the congruence $=_E$ of K^M spanned by the following two conditions:

$(CE1)$ if (a', b', b, a) is a pullback in K, then $S(b')S(a')^{\$} =_E S(a)^{\$}S(b)$;

$(CE2)$ if there exists a natural transformation $(1, f, 1) : (a, b) \dot{\to} (a', b')$, with f an E-morphism, then $S(b)S(a)^{\$} +_E S(b')S(a')^{\$}$.

3:3. For any category K, the symmetrization $S(=_E) :\ K \to K^M / =_E$ obviously depends on the choice of a subcategory E.

3:4. Proposition. (a) *if E =Iso(K), then the symmetrization induces by E is a relational embedding functor $G : K \to$Rel(K, E).*

(b) *Let E be a retractive subcategory of K. Then the equivalence relation $\sim$ defined in $R(A, B)$ by $(a, b) \sim (a', b')$ if and only if there exist E-morphisms e and e' such that $(a, b)e = (a', b')e'$, is compatible with composition and involution and it is spanned by $(CE2)$.*

(c) *For any retractive subcategory E of a category K, $G > S_E$, where $>$ is preorder relation between symmetrizations of K (induced by congruences of K^M).*

Proof. (a) Since E =Iso(K), any morphism in Rel(K, E) is an equivalence class $[f, g]$ of spans modulo izomorphism. In both cases, composition is given by pullback.

(b) $\sim$ is an equivalence relation compatible with composition and involution by $(E3)$. Let $=_E$ denote the relation spanned in K^I by $(CE2)$. It is implied by $\sim$ and so coincides with it. The second part is easy to prove by the fact that a retractive subcategory contains all isomorphisms.

(c) For any retractive subcategory E from a category K the corresponding symmetrization $S : K \to K^M/=_E$ may be defined. If the congruences of K^M are considered as "parts" of $K^M \times K^M$, a preorder relation, denoted by $>$, between symmetrizations of K is induced. Obviously, $E' \subset E$ implies $S_E < S_{E'}$, and, since for any retractive subcategory $E, E \subset$Iso(K), $S_E < S_{Iso(K)} = G$.

3:5. *Examples*. (a) (Conte [**2**], Kawahara [**3**]). The natural numbers N considered as a category with composition — ordinary multiplication of natural numbers, has pullbacks. For any, natural numbers m and n let j be the least common multiple of m and n. The square $(j/m, j/n, m.n)$ is a pullback. Obviously, N =Epi(N)=Mono(N) and so N is a retractive subcategory of N. The category of relations for N is of the form Rel(N, N) with arrows—rational numbers and composition—ordinary multiplicaton of rational numbers, as can easily be proved expressing the equivalence relation $=_N$ in the following form: $m = m'p$ and $n = n'p$ implies $m/n =_N m'/n'$. This shows that any map of this category $N^M/=_N$ can be represented uniqely as a span (m, n) and a natural transformation $e : (m, n) \dot{\to} (m', n')$ has components $1, p, 1$ respectively.

(b) Let K has finite products, E =Epi(K), and let F be a subcategory of K^I such that any functor (f, g) from K^I has a factorization $(f, g) = (f'e, g'e)$ unique up to isomorphism, where the functor (f', g') is from the subcategory F and e is from the category E. Then any map of Rel(K, E) has a uniquely determined representation given by a functor from F. Clearly, the morphisms of this category are the subobjects of the products.

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