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MAXIMAL CANONICAL GRAPHS WITH THREE NEGATIVE EIGENVALUES

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Abstract. A connected graph G is called canonical if no two of its nonadjacent vertices have the same neighbours in G. Let $C(3)$ be the class of all nonisomorphic canonical graphs with exactly 3 negative eigenvalues (including also their multiplicities). In this paper we prove that the class $C(3)$ contains exactly 32 maximal graphs with respect to relation to be induced subgraph. The orders of these graphs run over the set $\{9, 10, 11, 12, 13, 14\}$.

Results

A connected graph G from the class $C(3)$ is called minimal if each of its induced subraphs does not belong te this class, i. e. it is disconnected or not canonical or it has less than 3 negative eigenvalues. In [**2**] it is proved that the class $C(3)$ contains exactly 14 non-isomorphic minimal graphs, which we denote by $H_1, \dots, H_{14}$. The set of all these minimal graphs is denoted by $\mathcal{H}$. The graphs $H_i (i = 1, \dots, 14)$ are displayed in the following table according to the upper-diagonal form of their adjacency matrices. The first number in the bracket is the ordinal number of the graph H_i, the second one is the number $\mid H_i \mid$ of vertices in $\mid H_i \mid$, while the third one is the number of edges of H_i. The orders of graphs H_i are respectively 4,5 and 6.

The first graph in this list is the complete graph K_4, and the seventh one is the path P_6 on 6 vertices.

Obviously, each graph $G \in C(3)$ contains at least one the graphs $H \in \mathcal{H}$ as its minimal (induced) subgraph. But the corresponding minimal graph is not unique in the general case. The relation to be an induced subgraph is denoted by $\subseteq$.

The following two facts concerning the class $C(3)$ and minimal graphs have been also proved in [**4**].

Let $G \in C(3)$ and H be a minimal graph of G. Then:

(i)—each vertex $x \in V(G) - V(H)$ is adjacent to at least one vertex $y \in V(H)$;

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(ii)— no two vertices $x, y \in V(G) - V(H)$ can be adjacent to the same set of vertices from $V(H)$.

Table1. Minimal graphs with 3 negative eigenvalues

(01.4.6)	1	11	111		
(02.5.5)	1	11	001	0001	
(03.5.6)	1	11	011	0001	
(04.5.6)	1	11	001	0011	
(05.5.7)	1	10	110	1011	
(06.6.5)	1	01	001	0001	00100
(07.6.5)	1	01	001	0001	00001
(08.6.6)	1	01	001	0001	10001
(09.6.6)	1	01	101	0010	00010
(10.6.6)	1	11	100	0100	00100
(11.6.6)	1	01	100	0011	00001
(12.6.7)	1	10	011	0010	00011
(13.6.7)	1	01	101	0011	00100
(14.6.8)	1	01	100	0101	00111

Consequently, if S is an arbitrary nonempty set of vertices from $V(H)$ and $T_s = \{x \in V(G) - V(H) \mid x$ is adjacent to all vertices from S and nonadjacent to all other vertices from $V(H)\}$,

we have $| T_2 | \leq 1$. Thus, each of the sets $T_s(S \subseteq V(H))$ is empty or a singleton.

Using the method of forbidden subgraphs it is also shown in [**4**] which sets of the form $T_s(S \subseteq V(H))$ must be empty, for each particular graph $H \in \mathcal{H}$ and a graph $G \in C(3)$ such that $G \subseteq H$. Consequently, it is proved that $| G | \leq 18$ for each graph $G \in C(3)$.

Hence, to generate the class $C(3)$ or the set of all maximal graphs from this class, one can apply the method of extension of the graphs $H_i \in \mathcal{H}$ in each patricular case $i = 1, \ldots, 14$.

Namely, in each of the cases $H = H_i (i = 1, \ldots, 14)$ one can add to H_i all the possible sets of vertices of the form $T_{s_j}(S_j \subseteq V(H_i))$, and by computing the spectrum of the graph $H_i \cup \cup_j T_{s_j}$, one investigates all the possible cases related to induced subgraph $\cup_j T_{s_j}$ (connected or disconnected).

In view of the all previous results, this procedure is certainly finite. But although finite, it was of long duration and needed several hours of computer time.

Applying this procedure, we have generated all the graphs from the class $C(3)$ as well as all the maximal graphs from this class. The main result reads:

Theorem *The class $C(3)$ contains exactly 1800 nonisomorphic graphs.*

The same class also contains exactly 32 *maximal graphs which are presented in Table* 2.

We note that the first number in each line is the ordinal number of a maximal graph, the second number is the number of its vertices, and the last is the number of its edges.

Table 2. Maximal canonical graphs with 3 negative eigenvalues

(01.09.14)	1 01 001 0001 00100 010011 1001010 10101000
(02.09.14)	1 01 001 0001 00100 100100 1000110 10110001
(03.09.15)	1 11 001 0011 10010 010010 0101000 10001001
(04.09.15)	1 01 001 0001 00100 010011 1001010 10101100
(05.09.16)	1 11 001 0011 10010 010100 0100110 10001010
(06.09.17)	1 01 101 0010 00010 110011 1001100 01100101
(07.09.18)	1 11 001 0011 10010 010011 0101010 10001011
(08.10.19)	1 11 001 0001 10010 010100 0100110 00100011 100010101
(09.10.19)	1 01 001 0001 00100 101011 0100111 10010101 000011000
(10.10.21)	1 01 101 0010 00010 101011 0100110 01100110 100110001
(11.11.22)	1 10 110 1011 01100 100000 0011001 00111110 010010000 0000110000
(12.11.22)	1 01 001 0001 00100 011010 1010111 01001101 000011000 0001011000
(13.11.23)	1 01 001 0001 00100 011010 1010111 01001101 100001101 0000110000
(14.11.23)	1 01 001 0001 00100 011010 1000011 01001101 010100111 0000110001
(15.11.24)	1 11 111 0011 11001 100110 0110011 10100000 000100001 0100000010
(16.11.25)	1 10 110 1011 10000 010011 0000110 00010110 001101110 0110000110
(17.11.26)	1 10 110 1011 01100 100000 0011001 00011010 010010110 1110000111
(18.11.26)	1 10 110 1011 00001 001101 0100101 01100100 100001110 0011100011
(19.11.28)	1 11 001 0001 10010 001001 0101111 00001110 010101101 1001101101
(20.12.26)	1 11 111 0011 10100 010001 1100100 01100100 100000101 0010000000 00010100101
(21.12.26)	1 10 110 1011 00001 001101 0110010 00010001 010010101 1110010010 10000000000
(22.12.28)	1 11 111 0011 10100 011001 1100100 00010110 100110100 0010000011 01000100000
(23.12.30)	1 11 111 0011 11001 000101 1000000 10100110 011001111 0100000110 00100110000
(24.12.30)	1 11 111 0011 11001 000101 0100101 00100000 100110011 1010000000 01011000111
(25.12.30)	1 11 111 1000 01000 011010 1010011 00111111 000110110 0010010000 10010010000
(26.12.32)	1 11 111 0011 00100 010111 1001111 11001000 000100001 0100100101 10001010011
(27.12.36)	1 10 110 1011 01101 001101 0111000 01001011 100001111 1110001010 00011101011
(28.13.30)	1 11 111 0011 11001 100110 0110001 00010001 010100010 1000100000 01000000000 101001000101
(29.13.37)	1 11 111 0011 11001 000101 0100101 10010101 011010101 0101100001 10001011011 001000000010
(30.14.35)	1 11 111 1000 01000 011000 1010000 00100000 110000000 0001001110 10010011100 010100111000 0011100101010
(31.14.43)	1 11 111 0011 11001 000101 0100101 10010101 011010101 1000101101 00100000000 101010110100 0101010010101
(32.14.49)	1 11 111 0011 11001 000101 0100101 10010101 011010101 1000101101 10101011010 010101001011 0010011110101

Corollary 1. *A graph $G \in C(3)$ if and only if it is an induced subgraph of a graph from Table 2 and an induced overgraph of a graph $H \in \mathcal{H}$.*

Corollary 2. *For each graph $G \in C(3)$ we have $\mid G \mid \leq 14$.*

We indicate the possible minimal graphs of maximal graphs $M_1, \dots, M_{32}$ from Table 2. They are not unique in the general case. The notation $H_i \subseteq M_j$ for some $i \leq 14$ and some $j \leq 32$ will mean that H_i is isomorphic to the subgraph of M_j induced by its first $\mid H_i \mid$ vertices.

The following relations hold:

$H_1 \subseteq M_j (j = 15, 20, 22, 23, 24, 25, 26, 28, 29, 30, 31, 32)$,

$H_2 \subseteq M_j (j = 8, 19)$,

$H_4 \subseteq M_j (j = 3, 5, 7)$,

$H_5 \subseteq M_j (j = 11, 16, 17, 18, 21, 27)$,

$H_6 \subseteq M_j (j = 1, 2, 4, 9, 12, 13, 14)$,

$H_9 \subseteq M_j (j = 6, 10)$.

Finally, for each $m = 4, 5, \dots 14$, we denote by A_m the number of all non-isomorphic graphs from the class $C(3)$ whic have exactly m vertices. Then we have

$$A_4 = 1,\ \ A_5 = 6,\ \ A_6 = 43,\ \ A_7 = 170,\ \ A_8 - 372,\ \ A_9 = 499,$$
$$A_{10} = 404,\ \ A_{11} = 215,\ \ A_{12} = 72,\ \ A_{13} = 15,\ \ A_{14} = 3.$$

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