PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série tome 43 (57), 1988, 113-116

APPENDIX TO THE PAPER "EXISTENCE THEOREMS FOR L^p-SOLUTIONS OF INTEGRAL EQUATIONS IN BANACH SPACES"

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Let $D = [0, d]$ and let E be a real Banach space. Denote by $L^p(D, E)(p > 1)$ the space of all strongly measurable functions $u : D \to E$ such that

$$\|u\|_p = (\int_D \|u(t)\|^p dt)^{1/p} < \infty.$$

Consider the nonlinear Volterra integral equation

$$x(t) = g(t) + \int_0^t f(t, s, x(s))ds, \tag{1}$$

where

1° $g \in L^p(D, E)$;

2° $(t, s, x) \to f(t, s, x)$ is a function from $D^2 \times E$ into E which is continuous in x and strongly measurable in (t, s);

3° $\|f(t, s, x\| \le K(t, s)(m(s) + b\|x\|^{p/q})$ for $t, s \in D$ and $x \in E$, and

(i) $q > 1, b \ge 0$, $m \in L^q(D, R)$ and $m \ge 0$; let $r = q/(q - 1)$;

(ii) $(t, s) \to K(t, s)$ is a function from D^2 into R_+ such that $K(t, \cdot) \in L^r(D, R)$ for a. e. $t \in D$ and the function $t \to k(t) = \|K(t, \cdot)\|_r$ belongs to $L^p(D, R)$.

Let F be the mapping defined by

$$F(x)(t) = \int_0^t f(t, s, x(s))ds \qquad (x \in L^p(D, E), \;\; t \in D).$$

Assume that

4° $\lim\limits_{\tau \to 0} \sup\limits_{\|x\|_p \le \rho} \int_D \|F(x)(t + \tau) - F(x)(t)\| dt = 0$ for each $\varrho > 0$;

and

AMS Subject Classification (1980): Primary 45N05

5° $(t,s,u) \to h(t,s,u)$ is a nonnegative function defined for $0 \le s \le t \le d$, $u \ge 0$, satisfying the following conditions:

(i) for any nonnegative $u \in L^p(D,R)$ there exists the integral

$$\int_0^t h(t,s,u(s))ds \text{ for a. e. } \quad t \in D;$$

(ii) for any c, $0 < c \le d$, $u = 0$, a. e. is the only nonnegative function on $[0,c)$ which belongs to $L^p([0,c],R)$ and satisfies the inequality

$$u(t) \le 2\int_0^t h(t,s,u(s))ds \text{ almost everywhere on } [0,c].$$

Choose $\eta \in (0,1/2)$ and an interval $J = [0,a]$ in such a way that for ε, $0 \le \varepsilon \le \eta$, the maximal continuous solution z_ε of the integral equation

$$z(t) = \varepsilon + 2^{p-1}\int_0^t (\|g(s)\| + k(s)\|m\|_q + bk(s)z^{1/q}(s))^p ds$$

is defined on J and $z_\varepsilon(t) \le z_0(t) + 1$ for $t \in J$.

In [**3**] it was proved that if $1° - 5°$ hold and

$$\alpha(f(t,s,Z)) \le (t,s,\alpha(Z)) \tag{2}$$

for $0 \le s \le t \le d$ and for each bounded subset Z of E, where α denotes the Kuratowski measure of noncompactness, then the equation (1) has at least one solution $x \in L^p(J,E)$. Now we shall prove the following Aronszajn-type

THEOREM. *Under the above assumptions the set S of all solutions $x \in L^p(J,E)$ of (1) is a compact R_δ, i. e. S is homeomorphic to the intersection of a decreasing sequence of compact absolute retracts.*

Proof. Let $\varrho^p = \max_{t\in J} z_0(t) + 1$, $L^p = L^p(J,E)$, $B = \{x \in L^p : \|x\|_p \le \varrho\}$ and $U = \{x \in L^p : \|x\|_p \le \eta\}$. For any positive integer n and $x \in L^p$ put

$$F_n(x)(t) = \begin{cases} 0 & \text{if } 0 \le t \le a_n \\ \displaystyle\int_0^{t-a_n} f(t,s,x(s))ds & \text{if } a_n \le t \le a \end{cases}$$

where $a_n = a/n$. Then F_n is a continuous mapping $L^p \to L^p$ and

$$\|F_n(x)(t)\| \le k(t)\|m\|_q + bk(t)(\int_0^t \|x(s)\|^p ds)^{1/q}) \tag{3}$$

$$\|F(x)(t) - F_n(x)(t)\| \le k_n(t)(\|m\|_q + b(\int_0^t \|x(s)\|^p ds)^{1/q} \tag{4}$$

for $x \in L^p$, where

$$k_n(t) = \begin{cases} k(t) & \text{if} \quad 0 \le t \le a_n \\ \|K(t,\cdot)\chi_{[t-a_n t]}\|_r & \text{if} \quad a_n \le t \le a. \end{cases}$$

Put $G(x) = g + F(x)$ and $G_n(x) = g + F_n(x)$ for $x \in B$. Then G and G_n are continuous mapping of B into L^p and, by (4),

$$\lim_{n\to\infty} \|G(x) - G_n(x)\|_p = 0 \text{ uniformly in } x \in B. \tag{5}$$

Fix n. In the same way as in [**2**; p. 169] it can be shown that the mapping $I - G_n : B \to L^p$ is a homeomorhism into (I—the identity) and for a given $y \in U$ there exists $x_n \in L^p$ such that $x_n = y + g + F_n(x_n)$. In wiew of (3) we have

$$\|x_n(t)\| \le \|y(t)\| + \|g(t)\| + k(t)\|m\|_q + bk(t)(\int_0^t \|x_n(s)\|^p ds)^{1/q} \text{ for } t \in J.$$

Moreover, $2^{p-1}\int_0^t \|y(s)\|^p ds \le \eta$, because $\|y\|_p \le \eta \le 1/2$. Putting $w_n(t) = \int_0^t \|x_n(s)\|^p ds$, we obtain

$$w_n(t) \le \eta + 2^{p-1}\int_0^t (\|g(s)\| + k(s)\|m\|_q + bk(s)w_n^{1/q}(s))^p ds \text{ for } t \in J.$$

By the theorem on integral inequalities this implies

$$w_n(t) \le z_\eta(t) \le z_0(t) + 1 \le \varrho^p \quad \text{for } t \in J.$$

Hence $x_n \in B$. This shows that

$$U \subset (I - G_n)(B) \quad \text{for all } n. \tag{6}$$

Furthermore

$$(I - G)^{-1}(Y) \text{ is compact for each compact subset } Y \text{ of } L^p. \tag{7}$$

Indeed, let Y be a given compact subset of L^p and let (u_n) be a se-quence in $(I-G)^{-1}(Y)$. As $u_n - G(u_n) \in Y$, we can find a subsequencce (u_{n_j}) and $y \in Y$ such that $\lim_{j\to\infty} \|u_{n_j} - G(u_{n_j}) - y\|_p = 0$. By passing to a subsequence if necessary, we may assume that

$$\lim_{j\to\infty} (u_{n_j}(t) - G(u_{n_j}(t)) = y(t) \text{ for a. e. } t \in J.$$

Moreover, in view of (3) and the Eqoroff and Lusin theorems, for each $\varepsilon > 0$ there exist a closed subset J_ε and J and a number $M_\varepsilon > 0$ such that mes $(J\backslash J_\varepsilon) < \varepsilon$ and $\|u_{n_j t}\| \le M_\varepsilon$ for all j and $t \in J_\varepsilon$. Hence, putting $V = \{u_{n_j} : j = 1, 2, \dots\}$ and arguing similarly as in [**3**; p. 102], we conclude that V is relatively compact in L^p which proves (7).

From (5)–(7) it follows that the mapping I–G satisfies all assumptions of Thorem 7 of [**1**]. Therefore the set $(I-G)^{-1}(0)$ is a compact R_ϱ. On the other hand if $x \in S$, then analogously as for x_n in the proof of (6), it can be shown that $x \in B$. Consequently, $S = (I-G)^{-1}(0)$.

The author expresses his appreciation to the referee for his suggestions to amend an earlier version of the paper.

REFERENCES

[1] P. Browder, C. P. Gupta, *Topological degree and nonlinear mappings of analytical type in Banach spaces*, J. Math. Anal. Appl. **26** (1969), 390–402.

[2] S. Szufla, *On the existence of L^p-solutions of Volterra integral equations in Banach spaces*, Funkcialaj Ekvacioj **27** (1984), 157–172.

[3] S. Szufla, *Existence theorems for L^p-solutions of integral equations in Banach spaces*, Publ. Inst. Math. **40** (**54**) (1986), 99–105.

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(Received 08 06 1987)