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ON APPROXIMATION OF INTEGRABLE FUNCTIONS BY MODIFIED BERNSTEIN POLYNOMIALS

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Abstract. We introduce a class of positive linear operators defined for functions integrable on the simplex $\Delta = \{(x_1, x_2) : x_1 \geq 0, x_2 \geq 0, x_1 + x_2 \leq 1\}$ and study some approximations theorems on it.

1. Introduction. Recently Derriennic [2] gave some results on approximations of a function f integrable on [0,1] by the modified Bernstein polynomials of order n defined by

$$(B_n f)(x) = (n+1) \sum_{k=0}^{n} P_{nk}(x) \int_0^1 P_{nk}(t) f(t) dt, \tag{1.1}$$
$$P_{nk}(x) = \binom{n}{k} x^k (1-x)^{n-k}.$$

Denoting by $X = X(x_1, x2)$, a point in the simplex $\Delta = \{(x_1, x_2) : x_1 \geq 0, x_2 \geq 0, x_1 + x_2 \leq\}$ and writing $f(V)$ for $f(v_1, v_2)$, we define a new class of positive linear operators of order n by

$$(L_n f)(X) = \frac{(n+2)!}{n!} \sum_{k=0}^{n} \sum_{l=0}^{n-k} p_{nkl}(X) \iint_\Delta p_{nkl}(V) dv_1 dv_2, \tag{1.2}$$

where $p_{nkl}(X) = \binom{n}{k}\binom{n-k}{l} x_1^k x_2^l (1 - x_1 - x_2)^{n-k-l}$. In this paper we prove some results on approximation of a function f integrable on the simplex Δ by the polynomials (1.2).

2. Basic Propositions. PROPOSITION 1. *For $n \leq 1$, $(p, q = O, 1, 2 \ldots)$, one obtains*

$$(L_n v_1^p v_2^q)(X) = \frac{(n-2)!}{(n+p+g+2)!} \sum_{r=0}^{p} \binom{p!}{r} x_1^r \left[\sum_{l=0}^{q} \binom{q}{l} \frac{q!}{l!} x_2^l \cdots \frac{n!}{(n-r-l)!} \right].$$

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In particular we get

$$(L_n l)(X) = 1, \ \ (L_n v_i)(X) = (l + nx_i)/(n+3)$$
$$(L_n v_i^2)(X) = (2 + 4nx_i + n(n-1)x_i^2)/(n+3)(n+4); \ \ (i = 1, 2). \tag{2.3}$$

Proof. From definition (1.2), we get

$$I = (L_n v_1^p v_2^q(X) = \frac{(n+2)!}{n!}\sum_{k=0}^{n}\sum_{l=0}^{n-k} p_{nkl}(X)\iint_\Delta p_{nkl}(V)v_1^p v_2^q dv_1 dv_2$$

$$\frac{(n+2)!}{n!}\sum_{k=0}^{n}\sum_{l=0}^{n-k} p_{nkl}(X)\frac{n!}{k!l!(n-k-l)!}\iint_\Delta v_1^{k+p}v_2^{l+q}(1-v_1-v_2)^{n-k-l}dv_1 dv_2$$

Now the transformation $v_1 = t_1 t_2$, $v_2 = t_1(1-t_2)$, so that

$$dv_1, dv_2 = |\partial(v_1, v_2)/\partial(t_1, t_2)|dt_1 dt_2$$

reduces I to

$$\begin{aligned} I = {} & \frac{(n+2)!}{(n+p+q+2)!}\sum_{k=0}^{n}\sum_{l=0}^{n-k} p_{nkl}(X)\frac{n!}{k!(n-k-l)l!}\cdot \\ & \cdot\int_0^1\int_0^1 t_1^{k+l+p+q+1}(1-t_1)^{n-k-l}dt_1 t_2^{k+p}(1-t_2)^{k+q}dt_2, \\ = {} & \frac{(n+2)!}{(n+p+q+2)!}\sum_{k=0}^{n}\sum_{l=0}^{n-k} p_{nkl}(X)\frac{(k+p)!(l+q)!}{k!q!}, \\ = {} & ((n+2)!/(n+p+q+2)!)S \qquad \text{(say)} \end{aligned} \tag{2.4}$$

Now we use the expression

$$(\partial^{p+q}/\partial x_1^p \partial x_2^q)x_1^p x_2^q(x_1 + x_2 + y)^n \tag{2.5}$$

to evaluate (2.4) Clearly

$$\begin{aligned} & (\partial^{p+q}/\partial x_1^p \partial x_2^q))x_1^p x_2^q(x_1 + x_2 + y)^n \\ & = \sum_{k=0}^{n}\binom{n}{k}x_1^k\frac{(k+p)!}{k!}\Big[\frac{\partial^q}{\partial x_2^q}\{x_2^q(x_2+y)^{n-k}\}\Big] \\ & = \sum_{k=0}^{n}\binom{n}{k}x_1^k\frac{(k+p)!}{k!}\sum_{l=0}^{n-k}\binom{n-k}{l}x_2^l y^{n-k-l}\frac{(l+q)!}{l!}. \end{aligned} \tag{2.6}$$

Again differentiating (2.5) by Leibnitz theorem, we get

$$\begin{aligned} & (\partial^{p+q}/\partial x_1^p \partial x_2^q))x_1^p x_2^q(x_1 + x_2 + y)^n \\ & = \sum_{r=0}^{p}\binom{p}{r}\frac{p!}{r!}x_1^r\frac{n!}{(n-r)!}\Big\{\frac{\partial^q}{\partial x_2^q}x_2^q(x_1+x_2+y)^{n-r}\Big\} \\ & = \sum_{r=0}^{p}\binom{p}{r}\frac{p!}{r!}x_1^r\frac{n!}{(n-r)!}\Big[\sum_{l=0}^{q}\binom{q}{l}\frac{q!}{l!}x_2^l\frac{(n-r)!}{(n-r-0)!}(x_1+x_2+y)^{n-r-l}\Big] \end{aligned} \tag{2.7}$$

Now putting $y = l - x_1 - x_2$ in the expressions (2.6) and (2.7) and thus putting the value of S in (2.4), we get the required result.

PROPOSITION 2. *For $n \geq 1$ and $i, j \in \{1, 2\}$ we have*

$$L_n(v_i - x_i)(X) = 1 - 3x_i/(n+3), \tag{2.8}$$

$$L_n(v_i - x_i)^2(X) = 2nx_i(1 - xi)/(n+3)(n+4) + O(n^{-2}), \tag{2.9}$$

$$L_n(v_i - x_i)(v_j - x_j)(X) = -2x_i x_j n/(n+3)(n+4) + O(n^{-2}), \quad i \neq j \tag{2.10}$$

$$L_n(v_i - x_i)^2(v_j - x_j)(X) = O(n^{-2}), \quad i \neq j \tag{2.11}$$

$$L_n(v_i - x_i)^4(X) = O(n^{-2}) \tag{2.12}$$

Proof. Applying (2.1), by easy calculations we get the results (2.8) to (2.12).

PROPOSITION 3. *For $n \geq 1$ and $X \in \Delta$, we have*

$$L_n\left(\sum_{i=1}^{2}(v_i - x_i)^2\right)(X) \leq \frac{\max(8, n+2)}{(n+3)(n+4)} = Cn \quad (say) \tag{2.1.3}$$

Proof. We get from (2.3) that

$$\begin{aligned} &L_n\left(\sum_{i=1}^{2}(v_i - v_i)^2\right)(X) \\ &= [(2n-8)\{x_1(1-x_1) + x_2(1-x_2)\} + 4(1 + x_1^2 + x_2^2)]/(n+3)(n+4), \\ &= T/(n+3)(n+4) \end{aligned} \tag{2.14}$$

The maximum of the expression T on the simplex Δ for $n \geq 6$ occurs at $(1/2, 1/2)$ and it is $(n+2)$. The maximum value of T for $1 \leq n < 6$ is 8.

3. Main Results. THEOREM 1. *If f is an integrable and bounded function on the simplex Δ which has continuous derivatives up to the second order at a point $X \in \Delta$, then*

$$\lim_{n\to\infty} n\{(L_n f)(X) - f(X)\} = \sum_{i=1}^{2}\{(1 - 3x_i)f_i' + x_i(1 - xi)f_i''\} - 2x_1x_2 f''_{x_1x_2}. \tag{3.1}$$

Proof. Using Taylors's formula [3] for two variables, we write

$$\begin{aligned} &f(V = f(X) + \sum_{i=1}^{2}(v_i - x_i)f_i' + \\ &+ (1/2)\sum_{i,j=1}^{2}(v_i - x_i)(v_i - x_j)\{f_{ij}' + \alpha_{i,j}((v_i - x_i)(v_i - x_j), \ (v_j - x_j))\}, \end{aligned} \tag{3.2}$$

where $\alpha_{i,j}(0,0)=0$; $\alpha_{i,j}$ are integrable and bounded functions on the simplex Δ. Consequently for each $\varepsilon>0$, there exist positive numbers δ_1 and δ_j such that $|\alpha_i(v_i-x_i,v_j-x_j)|<\varepsilon$ when ever $|v_i-x_i|<\delta_i$, $|v_i-x_j|<\delta_j$, $1\le i,j\le 2$. Because of the boundedness of $\alpha_{i,j}$ on Δ, it follows that there exists $M>0$ such that $|\alpha_{i,j}(v_i-x_i,v_j-x_j)|<M$, $1\le i,j\le 2$. Now for every $\delta_i>0$, we define the function $\lambda_{\delta_l}(v_l)$ by

$$\begin{aligned}\lambda_{\delta_l}(v_l)&=1,\ \text{ when } |v_l-x_l|\ge\delta_l\\ &=0,\ \text{ when } |v_l-x_l|<\delta_l\end{aligned}$$

Thus for all $(v_i,v_i)\in\Delta$, the inequalities

$$|\alpha_{i,j}(v_i-x_i,v_j-x_j)\le\varepsilon+M\lambda_{\delta_i}(v_i)+M\lambda_{\delta_i}(v_i),\ (1\le i,j\le 2) \tag{3.3}$$

hold. Now in view of (2.8) to (2.12), (3.2) and (3.3) we get

$$(L_nf)(X)=f(X)+\sum_{i=1}^{2}\{(1-3x_i)f_i'+x_i(1-x_i)f_i''\}-2x_1x_2f''_{x_1x_2}+E_n(x_1,x_2),$$

where

$$\begin{aligned}E_n(x_1,x_2)=\frac{(n+2)!}{2n!}&\sum_{k=0}^{n}\sum_{l=0}^{n-k}p_{nkl}(X)\cdot\\ &\cdot\iint_\Delta p_{nkl}(V)\left\{\sum_{1\le i,j\le 2}(v_i-x_i)(v_i-x_j)\alpha_{i,j}(v_I-x_i,v_j-x_j)\right\}dv_1dv_2.\end{aligned}$$

It remains to show that $|nE_n|\to 0$ as $n\to\infty$. Clearly

$$\begin{aligned}|nEn(x_1,x_2)|\le(n/2)L_n[&\varepsilon(v_1-x_1)^2+\varepsilon(v_2-x_2)^2+2M(v_1-x_1)^4\delta_1^{-2}+\\ &+2M(v_2-x_2)^4\delta_2^{-2}+2(v_1-x_1)(v_2-x_2)\{\varepsilon+(v_1-x_1)^2\delta_1^{-2}+\\ &+(v_2-x_2)^2\delta_2^{-2}\}](X).\end{aligned}$$

Choosing $\delta_1=\delta_2=n^{-1/4}$ and taking the limit as $n\to\infty$, we get $|nEn(x_1,x_2)|\le 2\varepsilon$. Since ε is arbitrary, we have $|nEn|\to 0$ as $n\to\infty$.

THEOREM 2. *Let f be continuous in Δ and $\omega(f;1/\sqrt{n})$ be its modulus of continuity. Then for $n\ge 1$:*

$$\sup_{X\in\Delta}|(L_nf)(X)-f(X)|\le(1+nC_n)\cdot\omega(f;n^{-l/2}), \tag{3.4}$$

where C_n is given in (2.13).

Proof. We know that

$$(L_nf)(X)-f(X)=\frac{(n+2)!}{2n!}\sum_{k=0}^{n}\sum_{l=0}^{n-k}p_{nkl}(X)\iint_\Delta p_{nkl}(V)\{f(V)-f(X)\}dv_1dv_1.$$

Using the inequality

$$\omega(f;\delta) = \sup |f(V) - f(X)|, \quad \delta > 0, \quad \sqrt{(v_1 - x_1)^2 + (v_2 - x_2)^2} \le \delta$$

we get

$$|(L_n f)(X) - f(X)| \le \omega(f;\delta)\left\{ (L_n 1)(X) + \delta^{-2} L_n \left(\sum_{i=1}^{2} (v_i - x_i)^2 \right)(X) \right\}.$$

Taking $\delta = n$ and using (2.13), we get the required result.

it Remark. The result (3.4) can be obtained directly from Proposition 3 and Theorem 1 due to Censor [1].

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