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# INFLATION OF SEMIGROUPS

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**Abstract.** We introduce the concept of an $n$-inflation of a semigroup. In particular, for $n = 1$ we obtain the inflation introduced by Clifford [6], and for $n = 2$ the strong inflation introduced by Petrich [10]. We also characterize $n$-inflations of unions of groups, of semilattices of groups of unions of periodic groups, etc. In addition, we describe nilpotent semigroups of arbitrary nilpotency class.

## 1. Introduction and preliminaries

Let $S$ and $T$ be two disjoint semigroups and suppose that $T$ has a zero element. A semigroup $V$ is said to be an (ideal) *extension* of $S$ by $T$ if it contains $S$ as an ideal and the Rees factor semigroup $V/S$ is isomorphic to $T$. If, in addition, there is a partial homomorphism $\varphi : T\setminus 0 \to S$ such that for all $A, B \in T\setminus 0$ and $c, d \in S$:

$$A \circ B = \begin{cases} AB, & \text{if } AB \neq 0 \text{ in } T \\ \varphi(A)\varphi(B), & \text{if } AB = 0 \text{ in } T \end{cases}$$

$$A \circ c = \varphi(A)c, \ \ c \circ A = c\varphi(A), \qquad c \circ d = rd$$

we say that extension $V$ is determined by that partial homomorphism, [6].

Let $V$ be an extension of $S$. Than $V$ is a *retract extension* if there exists a homomorphism $\varphi$ of $V$ onto $S$ and $\varphi(x) = x$ for all $x \in S$. In this case we call $\varphi$ a *retraction*. Petrich [9] proved that an *extension $V$ of a semigroup $S$ by a semigroup $T$ with zero is determined by a partial homomorphism if and only if $V$ is a retract extension of $S$.* Here we give one more characterization of the retract extension.

Proposition 1.1. *Let $T$ be a semigroup. With each $a \in T$ associate a set $Y_a$ such that*

(1.1) $$a \in Y_a, \quad Y_a \cap Y_b = \emptyset \ \textit{ if } \ a \neq b.$$

*Let*

(1.2) $$\varphi^{(a,b)} : Y_a \times Y_b \to Y_{ab}$$

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$\varphi^{(a,b)}(x,b) = \varphi^{(a,b)}(a,y) = ab$ *for all* $x \in Y_a$ *and* $y \in Y_b$ *be functions for which*

$$\varphi^{(ab,c)}(\varphi^{(a,b)}(x,y),z) = \varphi^{(a,bc)}(x,\varphi^{(b,c)}(y,z)) \tag{1.3}$$

*and define a multiplication* $*$ *on* $S = \bigcup_{a\in T} Y_a$ *by:*

$$x * y = \varphi^{(a,b)}(x,y) \qquad \textit{if } x \in Y_a,\, y \in Y_b.$$

*Then* $(S,*)$ *is a semigroup and* $S$ *is a retract extension of* $T$*. Conversely, every retract extension* $S$ *of a semigroup* $T$ *can be so constructed.*

*Proof.* Suppose that $S$ fulfills the conditions of the proposition. Let $x \in Y_a$, $y \in Y_b$, $z \in T_c$. Then by (1.3) we have

$$\begin{aligned}(x*y)*z &= \varphi^{(a,b)}(x,y)*z = \varphi^{(ab,c)}(\varphi^{(a,b)}(x,y),z)\\ &= \varphi^{(a,ba)}(x,\varphi^{(b,c)}(y,z)) = x*\varphi^{(b,c)}(y,z)\\ &= x*(y*z).\end{aligned}$$

Hence $(S,*)$ is a semigroup. Define a mapping $\varphi : S \to T$ by $\varphi(Y_a) = a$. It is clear that $\varphi$ is onto and that $\varphi(a) = a$ for $a \in T$. Furthermore, for $x \in Y_a$, $y \in Y_b$ we have

$$\varphi(x*y) = \varphi(\varphi^{(a,b)}(x,y)) = ab = \varphi(x)\varphi(y).$$

Thus $\varphi$ is a homomorphism and by (1.2) $T$ is an ideal of $S$. Therefore, $S$ is a retract extension of $T$.

Conversely, let $S$ be a retract extension of $T$. Then there is a homomorphism $\varphi$ of $S$ onto $T$ such that $\varphi(a) = a$ for all $a \in T$. For $a \in T$ assume that $Y_a = \varphi^{-1}(a)$. Then $S = \bigcup_{a\in T} Y_a$ and for the sets $Y_a$ $(a \in T)$ the condition (1.1) is satisfied.

For any $x, y \in S$ there exist $a, b \in T$ such that $x \in Y_a$, $y \in Y_b$, so that $\varphi(x) = a$, $\varphi(y) = b$. From this it follows that

$$\varphi(xy) = \varphi(x)\varphi(y) = ab \in Y_{ab}$$

i.e. $xy \in Y_{ab}$. Hence there exist the functions

$$\varphi^{(a,b)} : Y_a \times Y_b \to Y_{ab}$$

and it is clear that for these functions (1.3) holds. Since $T$ is an ideal of $S$ we have (1.2).

Clifford [6, p. 98] gave a construction for a special retract extension of a semigroup, the so-called inflation of a semigroup. A semigroup $S$ is an *inflation* of a semigroup $T$ if $T$ is a subsemigroupof $S$ and there is a mapping $\varphi$ of $S$ onto $T$ such that $\varphi(x) = x$ for $x \in T$ and $xy = \varphi(x)\varphi(y)$ for $x, y \in S$. For further results concerning inflation of a semigroup, see [1], [3], [13], [14].

Petrich [10], [11], generalized Clifford's result introducing the notion of strong inflation.

Let $T$ be a semigroup. To each $a \in T$ we associate two sets $X_a$ and $Y_a$ having the following properties:

$$a \in X_a,\ X_a \cap X_b = Y_a \cap Y_b = \emptyset \text{ if } a \neq b; \quad X_a \cap Y_b = \emptyset \ (a, b \in T).$$

To every pair of elements $x \in Y_a, y \in Y_b$, we associate an element $\varphi^{(a,b)}(x,y) \in X_{ab}$. Now let $Z_a = X_a \cup Y_b$ and define a multiplication $*$ on $S = \bigcup_{a \in T} Z_a$ by: if $x \in Z_a, y \in Z_b$, then

$$x * y = \begin{cases} \varphi^{(a,b)}(x,y) & \text{if } x \in Y_a, y \in Y_b \\ ab & \text{otherwise.} \end{cases}$$

Then $S$ is a retract extension of $T$ and $S^3 \subset T$. Conversely, every retract extension $S$ of a semigroup $T$ such that $S^3 \subset T$ can be so constructed. Such a semigroup $S$ is called a strong inflation of a semigroup $S$. In particular for $T = 0$ nilpotent semigroups of nilpotency class $\leq 3$ are described, [12, p. 135]. Moreover, a semigroup $S$ in *n-nilpotent* if $S^n = 0$ $(n \in Z^+)$.

In this paper we introduce the notion of an $n$-inflation of a semigroup. For $n = 1$ we obtain the inflation and for $n = 2$ we obtain the strong inflation of semigroup. In Theorem 2.1. we describe an $n$-inflation of an arbitrary semigroup by means ot retraction. In section 2, also, a description of a strong $n$-inflation is given (Theorem 2.2.) and nilpotent semigroups of arbitrary nilpotency classes. In addition, we give characterizations of $n$-inflations of some special semigroups: unions of groups, semilattices of groups, unions of periodic groups and so on.

For undefined notions and notations we refer to [4], [6] and [12].

## 2. n-inflation of a semigroup

We introduce here the notion of an $n$-inflation of a semigroup.

LEMMA 2.1. *Let $T$ he a semigroup. To each $a \in T$ we associate a family of sets $X_i^a$ $(i = 1, 2, \dots, n)$ such that $a \in X_r^a$ for some $r \in \{1, 2, \dots, n\}$ and*

$$X_i^s \cap X_j^b = \emptyset \quad \text{if } i \neq j; X_i^a \cap X_j^b = \emptyset \quad \text{if } a \neq b. \tag{2.1}$$

*Let, for nonempty sets $X_i^a$ and $X_j^b$,*

$$\begin{gathered} \Phi_{(i,j)}^{(a,b)} : X_i^a \times X_j^b \to \bigcup_{\nu = i+j} X_\nu^{ab} \quad \text{if } i + j \leq n \\ \Phi_{(i,j)}^{(a,b)}(x,y) = ab \quad \text{if } i + j > n \\ \Phi_{(i,j)}^{(a,b)}(a,y) = \Phi_{(i,j)}^{(a,b)}(x,y) = ab \end{gathered} \tag{2.2}$$

*be functions for which:*

$$(\forall s \geq i + j)(\forall t \geq j + k)\Phi_{(s,k)}^{(ab,c)} \left(\Phi_{(i,j)}^{(a,b)}(x,y), z\right) = \Phi_{(i,t)}^{(a,bc)} \left(x, \Phi_{(j,k)}^{(b,c)}(y,z)\right) \tag{2.3}$$

*for all* $a, b, c \in T$, *where* $i+j \leq n$ *or* $j+k \leq n$ *or* $i+t \leq n$ *or* $s+k \leq n$.

*Let* $Y_a = \bigcup_{i=1}^{n} X_i^a$ *and define a multiplication* $*$ *on* $S = \bigcup_{a \in T} Y_a$ *by: for* $x \in Y_a, y \in Y_b$,

$$x * y = \Phi_{(i,j)}^{(a,b)}(x,y) \quad \textit{if } x \in X_i, y \in X_b, 1 \leq i, j \leq n$$

*Then* $(S,*)$ *is a semigroup.*

*Proof.* Let $x, y, z \in S$. Then there exist $a, b, c \in T$ such that $x \in Y_a$, $y \in Y_b$, $z \in Y_c$ i.e. $x \in X_i^a$, $y \in X_j^b, z \in X_k^c$ for some $1 \leq i, j, k \leq n$. Assume that $i+j \leq n$ and $j+k \leq n$. Then

$$\begin{aligned}
(x*y)*z &= \Phi_{(i,j)}^{(a,b)}(x,y)*z, \qquad && \Phi_{(i,j)}^{(a,b)}(x,y) \in X_s^{ab}, \quad i+j \leq s \leq n \\
&= \Phi_{(s,k)}^{(ab,c)}\left(\Phi_{(i,j)}^{(a,b)}(x,y), z\right) && \\
(x*y)*z &= x*\Phi_{(j,k)}^{(b,c)}(y,z), && \Phi_{(j,k)}^{(b,c)}(y,z) \in X_t^{bc}, \quad j+k \leq t \leq n \\
&= \Phi_{(i,t)}^{(a,bc)}\left(x\Phi_{(j,k)}^{(b,c)}(y,z)\right) &&
\end{aligned}$$

and by (2.3) we have associativity. In other cases it can be, in a similar way, proved that the associativity holds. Therefore $(S,*)$ is a semigroup.

*Definition* 3.1. The semigroup $S$ constructed in Lemma 2.1. is called an $n$-inflation of a semigroup T.

It is obvious that 1-inflation is the inflation, and that 2-inflation in the strong inflation. In those cases the condition (2.3) of Lemma 2.1 it not necessary.

The following theorem gives a characterization of an $n$-inflation of semigroups, which shows that here we have the case of retract extensions.

THEOREM 2.1. *A semigroup* $S$ *is an* $n$*-inflation of a semigroup* $T$ *if and only if* $S^{n+1} \subset T$ *and* $S$ *is a retract extension of* $T$.

*Proof.* Let $S$ be an $n$-inflation of a semigroup $T$. Then by (2.2) $T$ is an ideal of $S$. Assume $u \in S^{n+1}$, i.e. $u = s_1 * s_2 * \cdots * s_{n+1}$, $s_r \neq T$ $(r = 1, 2, \ldots, n+1)$. Let $s_r \in X_1^{a_r}$ where $a_r \in T$. Then

$$u = s_1 * s_2 * \cdots * s_{n+1} = \Phi_{(1,1)}^{(a_1,a_2)}(s_1, s_2) * s_3 * \cdots * s_{n+1}$$

If $2 > n$, then $\Phi_{(1,1)}^{(a_1,a_2)}(s_1, s_2) = u_1 \in T$, so $u \in T$.

If $2 \leq n$, then

$$\begin{aligned}
u &= u_1 * s_3 * \cdots * s_{n+1}, \quad u_1 \in X_{t_1}^{a_1 a_2}, \quad 2 \leq t \leq n. \\
&= \Phi_{(1,1)}^{(a_1 a_2, a_3)}(u_1, s_3) * s_4 * \cdots * s_{n+1}
\end{aligned}$$

If $t_1 + 1 > n$, then $\Phi_{(1,1)}^{(a_1 a_2, a_3)}(u_1, s_3) = u_2 \in T$, so $u \in T$.

If $t_1 + 1 \le n$ then $u = u_2 * s_3 * \cdots * s_{n+1}, \quad u_2 \in X_{t_2}^{a_1 a_2 a_3},\ 3 \le t_2 \le n.$

Continuing this procedure we have that: if $t_{n-2} + 1 > n$, then $\Phi_{(t_{n-2},1)}^{(a_1 \ldots, a_{n-1} a_n)} \cdot$ $\cdot (u_{n-2}, s_n) = u_{n-1} \in T$, so $u \in T$, and if $t_{n-2} + l \le n$, then $u = \Phi_{(t_n,1)}^{(a_1 \ldots, a_n a_{n+1})} \cdot$ $\cdot (u_{n-1}, s_{n+1}) \in T$, (since $n + 1 > n$).

In other cases ($_r \in X_{k_r}^{a_r},\ 1 < k_r \le n$) we have also that $u \in T$. Thus $S^{n+1} \subset T$.

Define a mapping $\Phi : S = \bigcup_{a \in T} Y_a \to T$ by $\Phi(Y_a) = a$. For any $x, y \in S$ there exist $a, b \in T$ such that $x \in Y_a, y \in Y_b$, i.e. $x \in X_i^a$, $y \in X_j^b$, for some $1 \le i, j \le n$. So

$$\Phi(x * y) = \Phi\left(\Phi_{(i,j)}^{(a,b)}(x, y)\right), \qquad \Phi_{(i,j)}^{(a,b)}(x, y) \in X_k^{ab} \subset Y_{ab}$$

for some $i + j \le k \le n$ if $i + j \le n$, and $\Phi(x * y) = ab$ if $i + j > n$. Now by the definition of $\Phi$ we have $\Phi(x * y) = ab = \Phi(x)\Phi(y)$. It is clear that $\Phi(x) = x$ for all $x \in T$. Therefore, $S$ is a retract extension of $T$.

Conversely, let $n$ be the smallest positive integer such that $S^{n+1} \subset T$ and let $\Phi$ be a retraction of $S$ onto $T$. An arbitrary $a \in T$ is in one of the following sets $S \setminus S^2,\ S^2 \setminus S^3, \ldots, S^{n-1} \setminus S^n,\ S^n$. For $a \in S^{n-r} \setminus S^{n-r+1}$ for some $0 \le r \le n-1$ we define the sets: $Y_a = \Phi^{-1}(a)$,

$$\begin{gathered}
X_1^a = Y_a \cap (S \setminus S^2) \\
X_2^a = Y_a \cap (S^2 \setminus S^3) \\
\vdots \\
X_{n-r-1}^a = Y_a \cap (S^{n-r-1} \setminus S^{n-r}) \\
X_{n-r}^a = Y_a \cap S^{n-} r \\
X_{n-r+1}^a = X_{n-r+2}^a = \cdots = X_n^a = \emptyset.
\end{gathered}$$

It is clear that the conditions (2.1) hold for every $X_i^a$ and $X_j^b$ $(1 \le i, j \le n)$.

If $a \in T$, then $Y_a = \bigcup_{i=1}^n X_i^a$ and so $S = \bigcup_{a \in T} Y_a$. For $x, y \in S$ there exist $a, b \in T$ such that $x \in Y_a, y \in Y_b$. So by Proposition 1.1. we have that

$$Y_a Y_b \subset Y_{ab} \tag{2.4}$$

Let $x \in X_i^a, y \in X_j^b, a \in S^{n-r} \setminus S^{n-r+1}, b \in S^{n-p} \setminus S^{n-p+1}$ where $0 \le r, p \le n-1$. Then

$x \in X_i^a = Y_a \cap (S^i \setminus S^{i+1})$ and $y \in Y_j^b = Y_b (S^j \setminus S^{j+1},\ 1 \le i \le n-r,\ 1 \le j \le n-p.$

Then $xy \in S^i S^j = S^{i+1}$ and if $i + j \le n$ we have that $xy \in \cap_{\nu=i+1}^n X_\nu^{ab}$. If $i + j > n$, then $xy = ab \in T$. For $x \in X_i^a$, $b \in T$ we have that $xb = ab,\ bx = ba$. In this way functions $\Phi_{(i,j)}^{(a,b)}$ from Lemma 2.1. are defined and the condition (2.3) holds.

*Definition* 2.2. If the first condition (2.2.) in the construction of an $n$-inflation in replaced by: For $1 \le i, j \le n$ let there exists a $k \in \{i + j, i + j + 1, \ldots, n\}$ and

$$\Phi_{(i,j)}^{(a,b)} : X_i^a \times X_j^b \to X_k^{ab}$$

then the semigroup $(S,*)$ is called the strong $n$-inflation of $T$.

The following theorem is proved similarly as the previous one.

THEOREM 2.2. *A semigroup $S$ is a strong $n$-inflation of a simigroup $T$ if and only if $S$ is an $n$-inflation of $T$ and the relation determined by the following partition $\{S\setminus S^2, S^2\setminus S^3,\dots,S^{n-1}\setminus S^n, S^n\}$ is a congruenre of $S$.*

*Example* 1. The semigroup $S$ given by the table 1 is a 4-inflation of $T=\{a,b\}$. Here we have $X_1^a=\{d,g\}$, $X_2^a=\{f\}$, $X_3=\{e\}$, $X_4^a=\{a,c\}$, $X_1^b=X_2^b=X_3^b=\emptyset$, $X_4^b=\{b\}$. $S$ is not strong 4-inflation of $T$. Since $d\cdot d=a\in X_4^a$ and $g\cdot g=f\in X_2^a$.

| 1 | $a$ | $b$ | $c$ | $d$ | $e$ | $f$ | $g$ |
|---|---|---|---|---|---|---|---|
| $a$ | $a$ | $b$ | $a$ | $a$ | $a$ | $a$ | $a$ |
| $b$ | $b$ | $a$ | $b$ | $b$ | $b$ | $b$ | $b$ |
| $c$ | $a$ | $b$ | $a$ | $a$ | $a$ | $a$ | $a$ |
| $d$ | $a$ | $b$ | $a$ | $a$ | $a$ | $a$ | $a$ |
| $e$ | $a$ | $b$ | $a$ | $a$ | $a$ | $a$ | $c$ |
| $f$ | $a$ | $b$ | $a$ | $a$ | $a$ | $c$ | $e$ |
| $g$ | $a$ | $b$ | $a$ | $a$ | $c$ | $e$ | $f$ |

| 2 | 0 | $a$ | $b$ | $c$ | $d$ |
|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 |
| $a$ | 0 | 0 | 0 | 0 | 0 |
| $b$ | 0 | 0 | 0 | $a$ | $a$ |
| $c$ | 0 | 0 | $a$ | $b$ | $b$ |
| $d$ | 0 | 0 | $a$ | $b$ | $b$ |

*Example* 2. The semigroup $S$ gives by the table 2 is a strong 3-inflation of $T=\{0\}$. Here we have $X_1^0=\{c,d\}$, $X_2^0=\{b\}$, $X_3^0=\{0,a\}$.

In particular, if $T=\{0\}$ then nilopent semigroups of nilpotency class $\le n+1$ are described by the following theorem which is directly proved by means of Theorem 2.1.

THEOREM 2.3. *Let $X_i,\ i=1,2,\dots,n$ be sets, let 0 be a fixcd element such that $\in X_n,\ X_i\cap X_j=\emptyset$ if $i\neq j$, and let*

$$\Phi_{(i,j)}: X_i\times X_j\to\bigcup_{v=i+j}^{n} X_v \ \text{ if } i+j\le n,\quad \Phi_{(i,j)}(x,y)=0 \ \text{ if } i+j>n$$

*be functions such that*

$$(\forall s\ge i+j)(\forall\ge j+k)\Phi_{(s,k)}\left(\Phi_{(i,j)}(x,y),z\right)=\Phi_{(i,t)}\left(x\Phi_{(j,k)}(y,z)\right)$$

*where $i+j\le n$ or $j+k\le n$ or $i+t\le n$ or $s+k\le n$. Define a multiplication $*$ on $S=\bigcup_{v=1}^n X_v$ by:*

$$x*y=\Phi_{(i,j)}(x,y)\quad \text{if}\quad x\in X_i,\, y\in X_j,\ 1\le i,j\le n.$$

*Then $(S,*)$ is a semigroup and $S^{n+1}=0$ and conversely, every nilpotent semigroup of nilpotency class $\le n+1$ can be so constructed.*

## 3. n-inflation of a union of groups

In the preceding section we considered $n$-inflations of a semigroup $T$ in the general case. In this sections we give characterization for those cases when $T$ is a union of groups, a semilattice of groups, and so on.

THEOREM 3.1. *The following conditions are equivalent on a semigroup S:*

(i) *$S$ is an n-inflation of a union of groups;*

(ii) $(\forall x, y \in S) xS^{n-1}y = x^2S^ny^2$*;*

(iii) $S^{n+1}$ *is a union of groups and*

$(\forall x_1, \dots, x_{n+1} \in S)(x_i^{n+1} \in G_{e_i} \Rightarrow x_1 \dots x_{n+1} = e_1x_1x_2 \dots x_{n+1}e_{n+1})$.

*Proof.* (i)⇒(ii). Let $S$ be an $n$-inflation of a union of groups $T$. Then $S^{n+1} = T$ is an ideal of $S$ and there exists a retraction $\varphi : S \to S^{n+1}$ (Theorem 2.1.). For any $x, x2, x3, \dots, x_n, y \in S$ there exists $e, f \in E(S)$ such that $\varphi(x) \in G_e$ and $\varphi(y) \in G_f$, so

$$\begin{aligned} xx_2x_3 \dots x_ny &= \varphi(x)\varphi(x_2)\varphi(x_3) \dots y\varphi(x_n)\varphi(y) \\ &= \varphi(x^{n+1})\varphi(x^{-1})\varphi(x_2) \dots y\varphi(x_n)\varphi(y^{-n})\varphi(y^{n+1}) \\ &\in x^{n+1}S^ny^{n+1} \subset x^2S^ny^2. \end{aligned}$$

Thus $xS^{n-1}y \subset x^2S^ny^2 \subset xS^{n-1}y$ and therefore (ii) holds.

(ii)⇒(iii). Let $x, y \in S$. Then

$$xS^{n-1}y = x^2S^ny^2 = (x^{n+1})^2S^n(y^{n+1})^2$$

so $x^{n+1} \in xS^{n-1}x = (x^{n+1})^2S^n(x^{n+1})^2$, i.e. $x^{n+1}$ is completely regular (Lemma I, 5.1. [3]). So $x^{n+1} \in G_e$ for some $e \in E(S)$. Let $u \in S^{n+1}$. Then

$$u = s_1s_1 \dots {}_{n+1} \in s_1S^{n-1}s_{n+1} = s_1^{n+1}S^nS_{n+1}^{n+1} = e_1s_1^{n+1}S^nS_{n+1}^{n+1}e_{n+1}$$

where $s_1^{n+1} \in G_{e_1}$, $s_{n+1}^{n+1} \in G_{en+1}$, and $e_1, e_{n+1} \in E(S)$. Thus $u = e_1u = ue_{n+1}$. This proves that the second condition of (iii) is fulfilled. Now

$$u = e_1u = e_1e_1 \dots e_1u \in e_1S^{n-1}u = e_1S^nu^2 \in Sn^2$$

and similarly $u \in u^2S$. So $u \in u^2Su^2$, i.e. $S^{n+1}$ is a union of groups (Lemma I 5.1. [3]).

(iii)⇒(i). Since $S^{n+1}$ is a union of groups we have that every regular element from $S$ is completely regular, i.e. $S$ is a $GV$-semigroup. Now by Theorem X.1.1. [3] (see also [15]) we have that $S$ is a semilattice $Y$ of semigroups $S_\alpha$, where $S_\alpha$ is a nil-extension of a completely simple semigroup $P_\alpha (\alpha \in Y)$. It is clear that $S_\alpha^{n+1} = P_\alpha$. Define a mapping $\varphi : S = \bigcup_{\alpha \in Y}, S_\alpha \to T = \bigcup_{\alpha \in Y} P_\alpha$ by

$$\varphi_\alpha = \varphi \restriction S_\alpha : S_\alpha \to P_\alpha; \qquad \varphi_\alpha(x_\alpha) = x_\alpha = x_\alpha e_\alpha, \text{ if } x_\alpha^{n+1} \in G_{e_\alpha}.$$

Then $\varphi_\alpha$ maps $S_\alpha$ onto $P_\alpha$ and $\varphi(x_\alpha) = x_\alpha$ for $x_\alpha \in P_\alpha$. Furthermore

$$\begin{aligned}
\varphi_\alpha(x_\alpha)\varphi_\beta(y_\beta) &= x_\alpha e_\alpha y_\beta e_\beta = e_\alpha x_\alpha y_\beta e_\beta && \text{(by Theorem I.4.3. [3]}\\
&= e_\alpha e_\alpha \dots e_\alpha x_\alpha y_\beta && \text{see, also [7])}\\
&= e_\alpha e_\alpha \dots e_\alpha x_\alpha y_\beta && \text{(by the hypothesis)}\\
&= e_\alpha e_\alpha \dots e_\alpha \dots x_\alpha y_\beta e_{\alpha\beta} && \text{(since } S \text{ is a semilattice } Y \text{ and}\\
&= e_\alpha x_\alpha y_\beta e_{\alpha\beta} e_{\alpha\beta} \dots e_{\alpha\beta} && \text{by the hypothesis)}\\
&= x_\alpha y_\beta e_{\alpha\beta} e_{\alpha\beta} \dots e_{\alpha\beta} && \text{(by the hypothesis)}\\
&= x_\alpha y_\beta e_{\alpha\beta}\\
&= \varphi_{\alpha\beta}(x_\alpha y_\beta)
\end{aligned}$$

for all $x_\alpha \in S_\alpha$, $y_\beta \in S_\beta$. Thus $S$ is an $n$-inflation of a semigroup $\bigcup_{\alpha\in Y} P_\alpha$, and $S_\alpha$ is an $n$-inflation of $P_\alpha$.

Corollary 3.1. *A seimigroup $S$ is an $n$-inflation of a completely simple semigroup if and only if $S^{n+1}$ is completely simple and the second condition of* (i) *of Theorem* 3.1 *holds.*

*Proof.* By the proof of Theorem 3.1.

A subset $B$ of a semigroup $S$ is two-sided $(m,n)$ pure if $B \cap x_1 \dots x_m S_{y-1} \dots y_n = x_1 \dots x_m B y_1 \dots y_n$ holds for every $x_1, \dots, x_m, y_1, \dots, y_n \in S$. A semigroup $S$ is two-sided $(m,n)$-pure if every bi-ideal of $S$ is a two-sided pure subset of $S$, [5].

Lemma 3.1. *Let $S$ be a semigroup. If $S^{n+1}$ is a semilattice of groups, then the idempotent elements of $S$ are central.*

*Proof.* By the hypothesis we have that $S$ is two-sided $(n-k,k)$-pure, $1 \le k \le n-1$, $n \ge 2$ [5, Theorem 1]. So $eSe\,(e \in E(S))$ is a two-sided $(n-k,k)$-pure bi-ideal of $S$. From this it follows that

$$xe \in xe \dots e \cdot eSe \cdot e \dots e = eSe \cap xe \dots eSe \dots e \subset eSe$$

for every $x \in S$. Thus $xe = eae$ for some $a \in S$ and similarly $ex = ebe$ for some $b \in S$. Now we have that

$$xe = eae = (ee)ae = e(eae) = e(xe) = (ex)e = (ebe)e = eb(ee) = ebe = ex.$$

Theorem 3.2. *The following conditions are equivalent on a semigroup $S$:*

(i) *$S$ is an $n$-inflation of a semilattice of groups,*

(ii) $(\forall x, y \in S)(xS^{n+1}y = y^2 S^n x)$,

(iii) *$S^{n+1}$ is a semilattice of groups.*

*Proof.* (i)$\Rightarrow$(iii) By Theorem 3.1 we have that $S^{n+1}$ is a union of groups and since the indempotents of $S$ are central we have that $S^{n+1}$ is a semilattice of groups.

(iii)$Rightarrow$(ii). For every $x, y \in S$ we have that $xS^{n-1}y = x^2S^ny^2 \subset x^2S^{n-1}y^2 \subset xS^{n-1}y$ [5, Theorem 1 ] i.e. $xS^{n-1}y = x^2S^{n-1}y^2 = x^2S^ny$. Thus

$$xS^{n-1}y = x^{n+1}S^my^{n+1} = (x^{n+1})^{-1}(x^{n+1})^2S^n(y^{n+1})^2(y^{n+1})^{-1},$$

since $x^{n+1} \in G_e$, $y^{n+1} \in G_f$ for some $e, f \in E(S)$. By Lemma 3.1 we have that the indempotents of $S$ are central, so

$$xS^{n-1}y = y^{n+1}(y^{n+1})^{-1}x^{n+1}S^ny^{n+1}(x^{n+1})^{-1}x^{n+1}$$

whence $xS^{n-1}y = y^2S^nx$.

(ii)$\Rightarrow$(iii). By the hypothesis we have that

$$xS^{n-1}y = y^2S^nx \subset y^2S^{n-1}x = x^2S^ny^2 \subset xS^{n-1}y$$

for every $x, y \in S$. So the condition (ii) of Theorem 3.1 holds. From this and Theorem 3.1. we have that $S^{n+1}$ is a union of groups. Since $S$ is weakly commutative, so is $S^{n+1}$. Thus $S^{n+1}$ is a semilattice of groups [2, Theorem,1.1].

(iii)$\Rightarrow$(i). By Lemma 3.1 the idempotents of $S$ are central. Thus $\varphi : S \to S^{n+1}$ defined by $\varphi(x) = xe$ if $x^{n+1} \in G_e$ is a retraction.

COROLLARY 3.2. *A Semigroup $S$ is an n-inflation of a group $T$ if and only if* $S^{n+1} = T$.

*Proof.* Trivial.

*Remark.* Semigroups from Theorem 3.2 are described in [5] by means ' of bi-ideals.

LEMMA 3.2. $S^{n+1}$ *is a union of periodic groups if and only if*

$$(\forall x_1, x_2, \dots, x_{n+1} \in S)(\exists m \in Z^+)x_1x_2\dots x_{n+1} = (x_1x_2\dots x_{n+1})^m.$$

*Proof.* Trivial.

COROLLARY 3.3. *A semigroup $S$ is an n-inflation of a semilattice if and only if*

$$(\forall x_1, x_2, \dots, x_{n+1} \in S)x_1x_2\dots x_{n+1} = (x_{n+l}x_2x_3\dots x_nx_1)^2$$

*Proof.* Follows by Theorem 3.2 and Lemma 3.2.

THEOREM 3.3. *A semigroup $S$ is an n-inflation of a union of periodic groups if and only if*

$$(\forall x_1, \dots, x_{n+1} \in S)(\exists m \in Z^+)x_1\dots x_{n+1} = x_1^{m+1}x_2\dots x_nx_{n+1}^{m+1}$$

*Proof.* Let $S$ be an $n$-inflation of a union of periodic groups. Then $x_i^{n+1} \in G_{e_1}$ for every $x_1, \dots, x_{n+1} \in S$, whence $x_i^m = e_i$ for some $m \in Z^+$, (since $G_{e_i}$ are periodic groups). Now by Theorem 3.1. we obtain

$$x_1x_2\dots x_{n+1} = e_1x_1x_2\dots x_{n+1} = e_1x_1x_2\dots x_{n+1}e_{n+1} = x_1^{m+1}x_2\dots x_nx_{n+1}^{m+1}.$$

Conversely, it is clear that $S$ is periodic. Assume $u \in S^{n+1}$. Then

$$u = x_1 x_2 \dots x_{n+1} = x_1^{m+1} x_2 \dots x_n x_{n+1}^{m+1} = x_1^{km+1} x_2 \dots x_n x_{n+1}^{km+1} \\ = e_1 x_1^{km+1} x_2 \dots x_n x_{n+1}^{km+1} e_{n+1}$$

where $x_1^{km} \in G_{e_1}$, $x_{n+1}^{km} \in G_{e_{n+1}}$ $(k \in Z^+)$, since $S$ is periodic. Hence, $u = e_1 x = y e_{n+1}$ for some $x, y \in S$. So

$$u = e_1 u = e e_1 \dots e_1 u = e_1 \dots e_1 u^{m+1} =^{m+1} .$$

Now by Lemma 3.2 we have that $S^{n+1}$ is a union of periodic groups. Since $u = e_1 u e_{n+1}$, and $x_i^{n+1} \in G_{e_1}$; for every $x_i^{n+1} \in S^{n+1}$ we have by Theorem 3.1 that the assertion of the theorem holds.

Corollary 3.3. *A semigroup $S$ is an $n$-inflation of a semilattice of periodic groups if and only if*

$$(\forall x_1, \dots, x_{n+1} \in S)(\exists m \in Z^+) x_1 \dots x_{n+1} = x_{n+1}^{m+1} x_2 \dots x_n x_1^{m+1}.$$

*Proof.* Follows by Theorem 3.2. and 3.3.

Following Nordahl, [8], we say that $S$ is an $E-m$ semigroup if the identity $(xy)^m = x^m y^m$ $(m \geq 2)$ holds in $S$.

Theorem 3.4. *The following conditions are equivalent on a semigroup $S$:*

(i) *$S$ is an $n$-inflation of a band;*

(i) *$S^{n+1}$ is a band and $S$ is an $E$-$(n+1)$ semigroup;*

(iii) *$S$ is a band $Y$ of nilpotent semigroups $S_\alpha$ of nilpotency class $\leq n$ and $Y \simeq E(S) = S^{n+1}$;*

(iv) $(\forall x_1, \dots x_{n+1} \in S) x_1 x_2 \dots x_{n+1} = x_1^2 x_2 \dots x_n x_{n+1}^2$;

*Proof.* (i)$\Rightarrow$(ii). Let $S$ be an $n$-inflation of a band $T$. Then by Theorem 2.1 $S^{n+1} \subseteq T$, $T$ is an ideal of $S$ and there is a retraction $\varphi : S \to T$. It is clear that $S^{n+1} = T$. Then for every $x, y \in S$,

$$(xy)^{n+1} = \varphi((xy)^{n+1}) = (\varphi(x)\varphi(y))^{n+1} = \varphi(x)\varphi(y) \\ = \varphi(x)^{n+1}\varphi(y)^{n+1} = \varphi(x^{n+1})\varphi(y^{n+1}) = x^{n+1}y^{n+1}.$$

Thus, $S$ is an $E$-$(n+1)$ semigroups.

(ii)$\Rightarrow$(i). Clearly $\varphi(x) = x^{n+1}$ is a retraction from $S$ onto $S^{n+1}$.

(ii)$\Rightarrow$(iii). Since $\varphi(x) = x^{n+1}$ is a homomorphism from $S$ onto the band $S^{n+1}$ we have that $\ker\varphi$ is a congruence $S$. Since $x(\ker\varphi)x^2$ for every $x \in S$ we have that $\ker\varphi$ is a band kongruence an the classes $\mod (\ker\varphi)$ are nilpotent semigroups of nilpotency class $\leq n$. Clearly $Y \simeq E(S) = S^{n+1}$.

(iii)$\Rightarrow$(ii). This implication follows immediately.

(i)$\Rightarrow$(iv). This equivalence follows by Theorem 3.3.

The following corollaries follow easily from the results already prove

COROLLARY 3.4. *The following conditions are equivalent on a semigroup S:*

(i) *$S$ is an n-inflation of a semilattice;*

(ii) *$S^{n+1}$ is a semilattice;*

(iii) $(\forall x_1, \dots, x_{n+1} \in S) x_1 x_2 \dots x_{n+1} = x_{n+1}^2 x_2 \dots x_n$.

COROLLARY 3.5. *A semigroup $S$ is an n-inflation of a rectangul band if and only if*

$$(\forall x_1, \dots, x_{n+3} \in S) x_1 x_2 \dots x_{n+3} = x_1 x_3 x_4 \dots x_{n+1} x_{n+3}$$

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