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ON THE QUASIASYMPTOTIC BEHAVIOUR OF THE STIELTJES TRANSFORMATION OF DISTRIBUTIONS

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Abstract. We obtain necessary and sufficient conditions under which the quasiasymptotic behaviour at infinity of an orginal determines the ordinary asymptotic behaviour at infinity of its Stieltjes transformation and conversely.

1. Introduction. The initial and final value Abelian theorems for the Stieltjes transformation of generalized functions were given in [4, 7, 8, 13, 15].

We have shown in [9] that under suitable conditions the Tauberian theorem for the classical Stieltjes transformation (5, p. 339) implies the corresponding Tauberian theorem for the distributional Stieltjes transformation. In investigations we have used the notion of the quasiasymptotic behaviour at infinity of tempered distributions with supports in $[0, \infty]$. For the distributional Stieltjes transformation this notion has some advantages in relation to the notions of the distributional asymptotic behaviour quoted in [4, 7, 8]. This was analyzed in [13].

In this paper we give (Theorem 5) sufficient conditions under which the behaviour at infinity of the Stieltjes transformation $(S_\varrho f)(x)$, $\varrho \in \mathbf{R} \setminus (-\mathbf{N}_0)$, $f \in I'(\varrho)$, (see Section 2) determines the quasiasymptotic behaviour of f at infinity. The given conditions are rather complicated and theoretical ones but it turns out that these conditions are also necessary (Theorem 4). In this way we obtain (Corollaries 6, 7) the necessary and sufficient conditions under which the quasiasymptotic behavour at infinity of an "original" determines the ordinary asymptotic behaviour at infinity of an "image" and conversely.

2. Notation and notions. As usual $\mathbf{C}$, $\mathbf{R}$, $\mathbf{Z}$ and $\mathbf{N}$ are the sets of complex, real, integer and natural numbers; $\mathbf{N}_0 = \mathbf{N} \cup \{0\}$.

We always denote by L a positive continuous function defined on $(0, \infty)$ such that

$$\lim_{t\to\infty} L(tx)/L(t) = 1 \quad \text{for every } x > 0.$$

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The function is called the slowly varying function at infinity [13]. Also, we shall assume that for some $\alpha > 0$, $L(x) \geq \alpha$, $x \in (0,\infty)$.

If for functions f and g, defined in a neighbourhood of ∞, there holds

$$\lim_{x\to\infty} f(x)/g(x) = l \neq 0,$$

then we say that f has the ordinary asymptotic behaviour as g and we introduce the notation $f \sim lg$, $x \to \infty$.

We denote by $\mathcal{S}'_+$ the space of all tempered distributions with supports in $[0,\infty)$.

Following [3] we say that an $f \in \mathcal{S}'_+$ has the quasiasymptotic at infinity related to $x^\nu L(x)$, $\nu \in \mathbf{R}$, if there exists the limit in the sense of $\mathcal{S}'$:

$$\lim_{k\to\infty} f(kx)/(k^\nu L(k)) = g(x), \quad \text{where} \quad \neq 0. \tag{1}$$

In this case we write for short: f has q.a.b. related to $x^\nu L(x)$ with the limit g. It is known ([3]) that $g(t)$ in (1) is equal to $A\ f_{\nu+1}(t)$ for some $A \neq 0$, where

$$f_{\alpha+1}(t) = H(t)t^\alpha/\Gamma(\alpha+1) \quad \text{for } \alpha > -1 \text{ and}$$
$$f_{\alpha+1}(t) = D^n f_{\alpha+n+1}(t) \quad \text{for } n+\alpha > -1,\ n \in \mathbf{N}.$$

As usual, H is the characteristic function of the interval $(0,\infty)$ and D stands for the distributional derivative.

We use in this article several properties of the quasiasymptotic behaviour of distributions which are proved in [3]. We summarize these properties in

THEOREM A [3]. (i) *If $f \in L^1_{\text{loc}}(\mathbf{R})$, supp $f \subset [0,\infty)$ and $f \sim f_{\nu+1}$, $x \to \infty$, for $\nu > -1$ then f has q.a.b. related to $x^\nu L(x)$ with the limit $g(x) = f_{\nu+1}(x)$.*

(ii) *A distribution $f \in CalL'_+$ has q.a.b. related to $x^\nu L(x)$ (with the limit $f_{\nu+1}$) iff there exists a natural number n, $n+\nu > 0$, such that for every $m \geq n$ there exists a continuous function $F_m(x)$ with the following properties:*

$$\mathit{supp}\, F_m \subset [0,\infty), \quad f = D^m F_m, \quad F_m \sim x^{\nu+m}L(x)/\Gamma(m+\nu+1), \quad x \to \infty.$$

If f has q.a.b. related to $x^\nu L(x)$, $\nu > n-1$, with the limit g, then $f^{(n)}$ has q.a.b. related to $x^{\nu-n}L(x)$ with the limit $g^{(n)}$, $n \in \mathbf{N}$.

(iii) *If $f \in \mathcal{S}'_+$ has q.a.b. related to $x^\nu L(x)$, $\nu > -1$, with the limit g, then for every $m > 0$, $x^m f(x)$ has q.a.b. related to $x^{m+\nu}L(x)$ with the limit $x^m g(x)$.*

Following Lavoine ane Misra [7, 8], we give the definition of the distributional Stieltjes transformation in the way presented in [12].

The space $J'(\varrho)$, $\varrho \in \mathbf{R}\setminus(-\mathbf{N}_0)$, is the space of all distributions with supports in $(0,\infty)$ such that $f \in J'(\varrho)$ iff there exists $k \in \mathbf{N}_0$ and $F \in L^1_{\text{loc}}(\mathbf{R})$, supp $F \subset [0,\infty)$, such that

$$\text{(a)}\ f = D^k F; \quad \text{(b)} \int_0^\infty |F(t)|(t+\beta)^{-\varrho-k}dt < \infty \ \text{ for } \beta > 0. \tag{2}$$

If instead of (b) we suppose that there exist $C = C(F)$ and $\varepsilon = \varepsilon(F) > 0$ such that

$$|F(x)| \leq C(1+x)^{\rho+k-1-\varepsilon} \quad \text{for } x \geq 0, \tag{c}$$

the corresponding space is denoted by $I'(\varrho)$. Obviously, $I'(\varrho) \subset J'(\varrho)$.

The Stieltjes transformation S_ϱ, $\varrho \in \mathbf{R} \setminus (-\mathbf{N}_0)$ of a distribution f from $J'(\varrho)$ with the properties given in (1) is a complex valued function $S_\varrho f$ given by

$$(S_\varrho f)(s) := (\varrho)_k \int_0^\infty F(t)(t+s)^{-\rho-k} dt, \quad s \in \mathbf{C} \setminus (-\infty, 0],$$

where $(\varrho)_k = \varrho(\varrho+1)\dots(\varrho+k-1)$, $k > 0$, and $(\varrho)_0 = 1$. It is proved in [8] that $S_\varrho f$ is a holomorphic function in $\mathbf{C} \setminus (-\infty, 0]$. If $f \in J'(\varrho + k)$, then $D^k f \in J'(\varrho)$ and

$$S_\varrho(D^k f) = (\varrho)_k (S_{\varrho+k} f).$$

Troughout the paper we shall assume that $f \in I'(\varrho)$, i.e. that (a) and (c) hold for f.

It is proved in [12] that for $f \in I'(\varrho)$

$$x^{[\varrho+k]} (S_\varrho f)_+ (x) \in L^1_{\text{loc}}(\mathbf{R}),$$

where

$$(S_\varrho f)_+ (x) = \begin{cases} 0, & x \leq 0 \\ (S_\varrho f)(x), & x > 0 \end{cases}$$

and $[\varrho + k]$ is the greatest integer not exceeding $\varrho + k$. If $(S_\varrho f)_+ \in L^1_{\text{loc}}(\mathbf{R})$, then we denote by $(\overline{S_\varrho f})_+$ the corresponding regular distribution. If $x^{l-1}(S_\varrho f)_+ (x) \notin L^1_{\text{loc}}(\mathbf{R})$ but $x^l (S_\varrho f)_+ (x) \in L^1_{\text{loc}}(\mathbf{R})$, where $l \in \mathbf{N}$, $l \leq [\varrho + k]$, then $(\overline{S_\varrho f})_+$ denotes the following regularization of $(\overline{S_\varrho f})_+$:

$$\begin{aligned} \langle (\overline{S_\varrho f})_+ (x),\ \phi(x) \rangle = \int_0^1 (S_\varrho f)_+ (x) \Big(\phi(x) - \phi(0) - \dots - \\ - \frac{x^{l-1}}{(l-1)!} \phi^{(l-1)}(0) \Big) dx + \int_1^\infty (S_\varrho f)(x) \phi(x) dx, \ \ \phi \in \mathcal{S}. \end{aligned} \tag{3}$$

Let $n \in \mathbf{N}$, $\varrho \in \mathbf{R} \setminus (-\mathbf{N}_0)$, $k \in \mathbf{N}_0$ be given numbers. We put

$$\mathcal{L}_{n,\varrho,k,x} = \frac{(-1)^{n+1} \Gamma(\varrho + k)}{(n+1)! \Gamma(n + \varrho + k)} D^{n+1} x^{2n+\varrho+k+2} D^{n+1}$$

and

$$L_{n,\varrho,k,x} = D^{k+1} \mathcal{L}_{n,\varrho,k+1,x} \quad ([12]).$$

Theorem B. [12] *Let $f \in I'(\varrho)$. For every $\phi \in \mathcal{S}$*

$$\lim_{n \to \infty} \langle L_{n,\varrho,k,x} (\overline{S_\varrho f})_+ (x),\ \phi(x) \rangle = \langle f(x),\ \phi(x) \rangle. \quad (k \textit{ is from } (c)).$$

We shall need the following three theorems.

THEOREM C. *Let* $a_{n,m}$, $n, m \in \mathbf{N}$, *be a matrix of complex numbers.*

(i) *If* $a_{n,m}$ *converges uniformly in* $m \in \mathbf{N}$ *to* a_m *as* $n \to \infty$ *and* $\lim_{m\to\infty} a_m$ *exists, then*

$$\lim_{n\to\infty}\lim_{m\to\infty} a_{n,m} = \lim_{m\to\infty}\lim_{n\to\infty} a_{n,m} = \lim_{\substack{n\to\infty\\ m\to\infty}} a_{n,m}.$$

(ii) *If* $\lim_{n\to\infty} a_{n,m}$ *exists for every* $m \in \mathbf{N}$, $\lim_{m\to\infty} a_{n,m}$ *exists for every* $n \in \mathbf{N}$, $\lim_{n,m\to\infty} a_{n,m}$ *exists, then* $a_{n,m}$ *converges uniformly in* $n \in \mathbf{N}$ *as* $m \to \infty$.

The assertions in Theorem C are well-known (see [2], for example).

THEOREM D. [1] *Let* g *be a locally integrable function on* $(0,\infty)$ *such that for some* p

$$\int_0^a x^{-p}|g(x)|dx \quad \textit{and} \quad \int_a^\infty x^p|g(x)|dx \ \textit{ converges } (a>0).$$

Then, there holds

$$\int_0^\infty g(x)L(\lambda x)dx \sim L(\lambda)\int_0^\infty g(x)dx, \ \ \lambda\to\infty.$$

THEOREM E. [13] (i) *Let* $f \in \mathcal{S}'_+$ *and* f *have q.a.b. related to* $x^a L(x)$ *with the limit* $Cf_{a+1}(x)$ $(C \neq 0)$. *Then for* $\varrho - 1 > a$, $\varrho \notin -\mathbf{N}_0$

$$(S_\varrho f)(x) \sim C\frac{\Gamma(\varrho-1-a)}{\Gamma(\varrho)}x^{-\varrho+1+a}L(x) \ \textit{ as } \ x\to\infty.$$

(ii) *If* $f \in J'(\varrho)$ *and* $(S_{\varrho+1}f)(x) \sim x^\alpha L(x)$, $x\to\infty$, *where* $\alpha < -1$, $\alpha \notin -\mathbf{N}$, *then* $(S_\varrho f)(x) = \varrho\int_x^\infty (S_{\varrho+1}f)(t)dt$ *and*

$$(S_\varrho f)(x) \sim \frac{\varrho}{-\alpha-1}x^{\alpha+1}L(x), \ \ x\to\infty.$$

The second part of Theorem E is not explicitely stated in [13] but it is proved during the course of proving the main theorem in [13].

3. Preliminary lemmas. LEMMA 1. *Let* $\varrho \in \mathbf{R}\setminus(-\mathbf{N}_0)$, $k \in \mathbf{N}_0$ *and* $\gamma \in \mathbf{C}$. *Then for every* $n \in \mathbf{N}$

$$\sum_{i=1}^{n+1}\binom{n+1}{i}(-1)^i(2n+\varrho+k+2)\dots(2n+\varrho+k+3-i)(2n+\varrho$$
$$+k+\gamma+2-i)\dots(\varrho+k+\gamma+i+1)+(2n+\varrho+k+\gamma+2)\dots(\varrho+k+\gamma+1)$$
$$= (-1)^n\gamma(1-\gamma)\dots(n-\gamma)(\varrho+k-\gamma+1)(n+1) \tag{4}$$

Proof. The proof follows by using the Leibniz formula

$$fD^m g = \sum_{i=0}^m(-1)^i\binom{m}{i}D^{m-i}(D^i fg), \quad f\in C^\infty, \ \ g\in\mathcal{D}'$$

in $D^{n+1}x^{2n+\varrho+k+2}D^{n+1}x^{\gamma}$ and by using the equality

$$D^{n+1}x^{2n+\varrho+k+2}D^{n+1}x^{\gamma} = (-1)^n\gamma(1-\gamma)\dots(n-\gamma)(\varrho+k+\gamma+1)_{n+1}x^{\varrho+k+\gamma}.$$

Now we give a lemma which is a consequence of [12, Lemma 1].

LEMMA 2. *Suppose that $f \in \mathcal{S}'_+$ and that f has q.a.b. related to $x^\nu L(x)$, where $\nu < \varrho - 1$. Then, there exist $k \in \mathbf{N}_0$, $k+\nu > 0$ and a continuous function F, supp $F \subset [0,\infty)$ such that $f = D^k F$ and for*

$$F_1 = \int_0^x F(t)dt, \;\; x \in \mathbf{R},$$

$\dfrac{\mathcal{L}_{n,\varrho,k+1,x}S_{\varrho+k+1}F_1(x) - F_1(x)}{x(1+x)^{\nu+k}}$ *converges uniformly to zero in* $(0,\infty)$.

Proof. By Theorem A (ii) we have that for some $k \in \mathbf{N}$, $k+\varrho > 0$, and some continuous function F, supp $F \subset [0,\infty)$, $f = D^kF$ and $F \sim x^{\nu+k}L(x)\,\Gamma(\nu+k+1)$, $x \to \infty$. Thus, for some $C > 0$

$$|F(x)| \le C(1+x)^{\nu+k}, \;\; x \ge 0, \quad \text{i.e. } f \in I'(\varrho).$$

By [12, Lemma 1] we obtain the assertion of Lemma 2. Namely, in the proof of Lemma 1 in [12] we have to use the inequality $|F(x)| < C(1+x)^{\nu+k}$, $x \ge 0$, instead of [12, (c)]. Let us notice that $\nu + k > 0$.

LEMMA 3. *Let $f \in I'(\varrho)$ and $(S_\varrho f)(x) \sim x^\nu L(x)$, $x \to \infty$, $\nu > -1$. Then $(\overline{S_\varrho f})_+(x)$ has q.a.b. to $x^\nu L(x)$ as well. ($(\overline{S_\varrho f})$ is defined by* (3).)

Proof. We have ($\phi \in \mathcal{S}$)

$$\begin{aligned}\frac{1}{k^\nu L(k)}\langle\overline{(S_{\varrho f})_+}(kx),\ \phi(x)\rangle &= \frac{1}{k^{\nu+1}L(k)}\left\langle\overline{(S_{\varrho f})_+}(x),\ \phi\left(\frac{x}{k}\right)\right\rangle\\ &= \frac{1}{k^{\nu+1}L(k)}\int_0^1 (S_\varrho f)(\phi(x/k) - \phi(0) - \cdots\\ &\qquad \cdots - \left(\frac{x}{k}\right)^{l-1}\left(\frac{\phi^{(l-1)}(0)}{(l-1)!}\right)dx + \frac{1}{k^{\nu+1}L(k)}\int_1^\infty (S_\varrho f)(x)\phi\left(\frac{x}{k}\right)dx.\end{aligned} \tag{5}$$

Since the first part on the right hand side of (5) converges to zero as $k \to \infty$ we have to prove that

$$\frac{1}{k^\nu L(k)}\int_{1/k}^\infty (S_\varrho f)(kx)\phi(x)dx \to \int_0^\infty x^\nu L(x)\phi(x)dx \quad \text{as } x \to \infty.$$

Recall that $(S_\varrho f)(x) \sim x^\nu L(x)$, $x \to \infty$. This implies that for a given $\varepsilon > 0$ there exists $x_0 > 0$ such that

$$|(S_\varrho f)(x) - x^\nu L(x)| \le \varepsilon x^\nu L(x), \quad x \ge x_0 > 1. \tag{6}$$

We use the following decomposition:

$$
\begin{aligned}
&\frac{1}{k^\nu L(k)}\int_{1/k}^{\infty}(S_\varrho f)(kx)\phi(x)dx\\
(7)\qquad &= \frac{1}{k^\nu L(k)}\int_{1/k}^{x_0/k}(S_\varrho f)(kx)\phi(x)dx+\int_{x_0/k}^{\infty}(S_\varrho f)(kx)\phi(x)dx.
\end{aligned}
$$

The first member on the right hand side of (7) tends to zero when $k\to\infty$ because

$$
(8)\qquad \frac{1}{k^\nu L(k)}\int_{1/k}^{x_0/k}|(S_\varrho f)(kx)\phi(x)|dx \le \frac{M}{k^\nu L(k)}\max_R\{|\phi(x)|\}\frac{x_0-1}{k}
$$

where

$$
M=\max\{|(S_\varrho f)(x)|;\ \ 1\le x\le x_0\}.
$$

Also, one can prove easily that for a given $x_0>1$

$$
(9)\qquad \frac{1}{L(k)}\int_0^{x_0/k}|x^\nu L(kx)\phi(x)|dx\to 0\quad \text{as }\ k\to\infty.
$$

Now by (6), (8), (9), Theorem D and

$$
\begin{aligned}
&\left|\frac{1}{k^\nu L(k)}\int_{1/k}^{\infty}(S_\varrho f)(kx)\phi(x)dx-\frac{1}{k^\nu L(k)}\int_0^{\infty}(kx)^\nu L(kx)\phi(x)dx\right|\\
&\le\frac{1}{k^\nu L(k)}\int_{1/k}^{x_0/k}|(S_\varrho f)(kx)\phi(x)|dx+\frac{1}{k^\nu L(k)}\int_0^{x_0/k}|(kx)^\nu L(kx)\phi(x)|dx\\
&\qquad+\frac{1}{k^\nu L(k)}\int_{x_0/k}^{\infty}|(S_\varrho f)(kx)-(kx)^\nu L(kx)||\phi(x)|dx
\end{aligned}
$$

we obtain the assertion

4. Main results. First we prove the following theorem.

THEOREM 4. *If $f\in\mathcal{S}'_+$ and f has q.a.b. related to $x^\nu L(x)$, $\varrho-2<\nu<\varrho-1$, $\varrho\in\mathbf{R}\setminus(-\mathbf{N})$, then the double sequence*

$$
(10)\qquad \left\{\left\langle\frac{\overline{(S_{\varrho+k+1}F_1)_+}(x)}{m^{\nu+k+1}L(m)},\ \mathcal{L}_{n,\varrho,k+1,x}\phi^{(k+1)}(x)\right\rangle;\ m,\,n\in\mathbf{N}\right\},\quad \phi\in\mathcal{S},
$$

converges uniformly in $n\in\mathbf{N}$, as $m\to\infty$, where $k\in\mathbf{N}$, $k+\nu>0$, and F_1 are defined in the proof of Lemma 2.

Proof. Let

$$
a_{n,m}=\left\langle\frac{L_{n,\varrho,k,x}\overline{(S_\varrho f)}_+(mx)}{m^\nu L(m)},\ \phi(x)\right\rangle.
$$

Since

$$a_{n,m} = (-1)^{k+1}(\varrho)_{k+1}\left\langle \frac{\overline{(S_{\varrho+k+1}F_1)}_+(mx)}{m^{\nu+k+1}L(m)},\ \mathcal{L}_{n,\varrho,k+1,x}\phi^{(k+1)}(x)\right\rangle$$

we have to prove that $a_{n,m}$ converges uniformly in $n \in \mathbf{N}$ as $m \to \infty$. First we shall prove that the conditions of Theorem C (i) are satisfied. Then Theorem 4 will follow from Theorem C (ii).

Theorem B implies

$$a_{n,m} \to a_m = \left\langle \frac{f(mx)}{m^{\nu}L(m)},\ \phi(x)\right\rangle,\ n \to \infty,\ m \in \mathbf{N}.$$

Since

$$\begin{aligned}a_{n,m} - a_m &= (-1)^{k+1}(\varrho)_{k+1}\int_0^\infty \frac{\mathcal{L}_{n,\varrho,k+1,x}(S_{\varrho+k+1}F_1)(mx) - F_1(mx)}{m^{\nu+k+1}L(m)}\phi^{(k+1)}(x)dx\\ &= (-1)^{k+1}(\varrho)_{k+1}\int_0^\infty \frac{\mathcal{L}_{n,\varrho,k+1,x}(S_{\varrho+k+1}F_1)(mx)(mx)(1+mx)^{\nu+k}}{(mx)(1+mx)^{\nu+k}m^{\nu+k+1}L(m)}\phi^{(k+1)}(x)dx,\end{aligned}$$

from Lemma 2 and

$$\frac{mx(1+mx)^{\nu+k}}{m^{\nu+k+1}L(m)} \le 2\frac{mx(1+(mx)^{\nu+k})}{\alpha m^{\nu+k+1}} \le \frac{2x+2x^{\nu+k+1}}{\alpha},\ \ x \ge 0$$

we obtain that

$$a_{n,m} - a_m \to 0 \quad \text{uniformly in } m \in \mathbf{N} \text{ as } n \to \infty.$$

By Theorem E (i) we have $(S_\varphi f)(x) \sim \frac{\Gamma(\varrho-\nu-1)}{\Gamma(\varrho)}x^{\nu-\varrho+1}L(x)$, $x \to \infty$. Since $-1 < \nu - \varrho + 1 < 0$, we have by Lemma 3 that $\overline{(S_\varrho f)}_+(x)$ has q.a.b. related to $x^{\nu-\varrho+1}L(x)$ with the limit $\frac{\Gamma(\varrho-\nu-1)}{\Gamma(\varrho)}x^{\nu-\varrho+1}$.

By Leibniz formula we have

$$\begin{aligned}&D^{n+1}x^{2n+\varrho+k+2}D^{n+1}(S_{\varrho+k+1}F_1)(x)\\ &\quad= D^{n+1}\sum_{i=0}^{n+1}(-1)^i\binom{n+1}{i}((x^{2n+\varrho+k+2})^{(i)}(S_{\varrho+k+1}F_1)(x))^{(n+1-i)}\\ &\quad= \sum_{i=1}^{n+1}(-1)^i\binom{n+1}{i}(2n+\varrho+k+2)\dots(2n+\varrho+k+3-i)(x^{2n+\varrho+k+2-i}\\ &\qquad\cdot(S_{\varrho+k+1}F_1)(x))^{(2n+2-i)} + (x^{2n+\varrho+k+2})(S_{\varrho+k+1}F_1(x))^{(2n+2)}.\end{aligned}$$

Let $\gamma = \nu - \varrho + 1$. We have

$$x^{2n+\varrho+k+2-i}(S_{\varrho+k+1}F_1)(x) \text{ has q.a.b. related to } x^{2n+\varrho+k+\gamma-i}L(x)$$

with the limit

$$\Gamma(-\gamma)x^{2n+\varrho+k+\gamma+2-i}/(\Gamma(\varrho)(\varrho)_{k+1}).$$

Lemma 1 and Theorem A (iii) imply that $D^{n+1}x^{2n+\varrho+k+2}(S_{\varrho+k+1}F_1)(x)$ has q.a.b. related to $x^{\varrho+k+\gamma}L(x)$ with the limit

$$(-1)^n\gamma(1-\gamma)\dots(n-\gamma)(\varrho+k+\gamma+1)_{n+1}\frac{\Gamma(-\gamma)}{\Gamma(\varrho)(\varrho)_{k+1}}x^{\gamma+k+\varrho}.$$

Thus we obtain

$$\begin{aligned}&\lim_{n\to\infty}\lim_{m\to\infty}(-1)^{(k+1)}(\varrho)_{k+1}\left\langle\frac{\mathcal{L}_{n,\varrho,k+1,x}\overline{(S_{\varrho+k+1}F_1)}_+(mx)}{m^{\nu+k+1}L(m)},\ \phi^{(k+1)}(x)\right\rangle\\&=-\lim_{n\to\infty}\frac{(-1)^{k+1}\Gamma(\varrho+k+1)}{(n+1)!\Gamma(n+\varrho+k+1)}\frac{\Gamma(n-\gamma+1)}{\Gamma(-\gamma)}\frac{\Gamma(\varrho+k+\gamma+n+2)}{\Gamma(\varrho+k+\gamma+1)}\frac{\Gamma(-\gamma)}{\Gamma(\varrho)}\\&\quad\langle x^{\nu+k+1},\ \phi^{(k+1)}(x)\rangle.\end{aligned}$$

To prove that the last limit exists we have to use the Stirling formula $\Gamma(s+1)\sim\sqrt{2\pi}e^{-s}s^{s+1/2}$, $s\to\infty$.

Thus, for the double sequence $a_{n,m}$ Theorem C (i) holds and Theorem C (ii) implies the assertion.

TAUBERIAN THEOREM 5. *Let $f\in I'_\varrho$ and let $\overline{(S_\varrho f)}_+(x)$ have q.a.b. related to $x^\alpha L(x)$ where $-1<\alpha<0$. If for any $\phi\in\mathcal{S}$ the double sequence* (10) *converges uniformly in $n\in\mathbf{N}$ as $m\to\infty$, then f has q.a.b. related to $x^{\alpha+\varrho-1}L(x)$.*

Proof. Follows from Theorem C (i) with $a_{n,m}$ defined as in the proof of Theorem 5.

Theorem 4 and 5 imply

COROLLARY 6. *Let $f\in I'(\varrho)$, $\varrho\in\mathbf{R}\setminus(-|boldN_0)$ and $-1<\alpha<0$. The conditions*

(i) *f has q.a.b. related to $x^{\alpha+\varrho-1}L(x)$;*

(ii) *$\overline{(S_\varrho f)}_+(x)$ has q.a.b. related to $x^\alpha L(x)$;*

are equivalent iff the double sequence (10) *converges uniformly in $n\in\mathbf{N}$ as $m\to\infty$.*

Also, Theorems 4, 5 and Theorem E (ii) imply

COROLLARY 7. *Let $f\in I'(\varrho-p)$ $\varrho\in\mathbf{R}\setminus(-\mathbf{N}_0)$, $p\in\mathbf{N}$ and $-1<\alpha<0$. The conditions*

(i) *f has q.a.b. related to $x^{\alpha-p+\varrho-1}L(x)$,*

(ii) *$(S_\varrho f)(x)\sim x^{\alpha-p}L(x),\ x\to\infty$,*

are equivalent iff for every $\phi\in\mathcal{S}$ the double sequence

$$\left\langle\frac{L_{n,\varrho-p,k,x}\overline{(S_{\varrho-p}f)}_+(mx)}{m^{\alpha-p+\varrho-1}L(m)},\ \phi(x)\right\rangle,\quad m,n\in\mathbf{N}\tag{11}$$

converges uniformly in $n\in\mathbf{N}$ as $m\to\infty$.

At the end we give a "classical" result on the classical Stieltjes transformation of functions.

COROLLARY 8. *Let f be a locally integrable function such that $f \in I'(\varrho - p)$, $\varrho - p > 0$, $p \in \mathbf{N}$ and let $-1 < \alpha < 0$. If $(S_\varrho f)(x) \sim x^{\alpha-p}L(x)$, $x \to \infty$, and the double sequence* (11) *converges uniformly in $n \in \mathbf{N}$ as $m \to \infty$, then f is the s-th classical derivative $(s \in \mathbf{N}_0)$ of some function $F \in L^1_{\mathrm{loc}}(\mathbf{R})$ for which*

$$F \sim Cx^{\alpha-p+\varrho+s-1}L(x), \quad x \to \infty \ \ (C \neq 0)$$

holds.

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