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EXISTENCE THEOREMS FOR L^p – SOLUTIONS OF INTEGRAL EQUATIONS IN BANACH SPACES

Stanislaw Szufla

Abstract. We study the integral equation $x = F(x)$ in a Banach space E, where $F(x)(t) = \int_D f(t, s, x(s))ds$ and f satisfies usual conditions which guarantee that F continuously maps the space $L^P(D, E)$ into itself. We show that if f satisfies a Kamke condition with respect to the Kuratowski measure of noncompactness, then the above equation has a solution in $L^P(D, E)$.

1. Introduction. Let $D = [0, d]$ be a compact interval in R and let E be a real Banach space. Denote by $L^p(D, E)(p > 1)$ the space of all strongly measurable functions $u : D \to E$ with $\int_D \|u(t)\|^p dt < \infty$, provided with the norm $\|u_p\| = (\int_D \|u(t)\|^p dt)^{1/p}$.

In this paper we consider the Urysohn integral equation

$$x(t) = g(t) + \lambda \int_D f(t, s, x(s))ds \tag{1}$$

and the Volterra integral equation

$$x(t) = g(t) + \int_0^t f(t, s, x(s))ds. \tag{2}$$

Throughout this paper we shall assume that

1° $g \in L^p(D, E)$.

2° $(t, s, x) \to f(t, s, x)$ is a function from $D^2 \times E$ into E which is continuous in x for a.e. $t, s \in D$ and strongly measurable in (t, s) for each $x \in E$.

3° $\|f(t, s, x)\| \leq K(t, s)(a(s) + b\|x\|^{p/q})$ for $t, s \in D$ and $x \in E$ where

(i) $q > 1$, $b \geq 0$, $a \in L^q(D, R)$ and $a \geq 0$; let $r = q/(q - 1)$;

(ii) $(t, s) \to K(t, s)$ is a function from D^2 into R_+ such that $K(t, \cdot) \in L^r(D, R)$ for a.e. $t \in D$ and the function $t \to k(t) = \|K(t, \cdot)\|_r$, belongs to $L^p(D, R)$.

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In contrast to the case $E = R^n$, the conditions 1°–3° are not sufficient for the existence of a solution $x \in L^p(D, E)$ of (1) or (2) when E is infinite dimensional. Therefore one has imposed additional conditions on f. In [5] it was proved that the integral equation

$$x(t) = g(t) + \int_0^t Q(t,s)\varphi(s, x(s))ds \tag{3}$$

has an L^p – solution whenever φp satisfies a Hölder condition expressed in terms of Kuratowski's measure of noncompactness. Now we shall show that similar theorems are valid also for the equations (1) and (2). Let us remark that the existence proofs given below differ totally from those in [5].

2. Measures of noncompactness. Let α and α_p denote the Kuratowski measures of noncompactness in E and $L^p(D, E)$, respectively. For a given set V of functions from D into E we define a function ν by $\nu(t) = \alpha(V(t))$ for $t \in D$ (under the convention that $\alpha(X) = \infty$ if X is unbounded), where $V(t) = \{x(t) : x \in V\}$. Recently Heinz [2] proved the following important

THEOREM 1. *Let V be a countable set of strongly measurable functions $D \to E$ such that there exists $\mu \in L^1(D, R)$ such that $\|x(t)\| \le \mu(t)$ for all $x \in V$ and $t \in D$. Then the corresponding function ν is integrable and*

$$\alpha\left(\left\{\int_D x(t)dt : x \in V\right\}\right) \le 2\int_D \nu(t)dt.$$

Now we shall prove a theorem clarifying the relation between α and α_p.

THEOREM 2. *Let V be a countable set of strongly measurable functions $D \to E$ such that*

(i) there exists $\mu \in L^p(D, R)$ such that $\|x(t)\| \le \mu(t)$ for all $x \in V$ and $t \in D$;

(ii) $\lim_{h\to 0} \sup_{x\in V} \int_D \|x(t+h) - x(t)\|^p dt = 0$. Then $\alpha_p(V) \le 2\left(\int_D \nu^p(t)dt^{1/p}\right)$.

Proof. Without loss of generality we may assume that all functions from $L^p(D, E)$ are extended to R by putting $x(t) = 0$ outside D. For any positive number h put $V_h = \{x_h : x \in V\}$, where

$$x_h(t) = \frac{1}{2h}\int_{t-h}^{t+h} x(s)ds \quad \text{for} \quad t \in D.$$

It is well known that under the assumptions (i) and (ii) the set V_h is equicontinuous and uniformly bounded. Moreover,

$$\lim_{h\to 0} \|x - x_h\|_p = 0 \quad \text{uniformly in } x \in V.$$

This implies that

$$\alpha_p(V) = \lim_{h\to 0} \alpha_p(V_h) \tag{4}$$

and, by Lemma 3 of [5],

$$\alpha_p(V_h) \leq \left(\int_D (\alpha(V_h(t)))^p dt\right)^{1/p}.$$

On the other hand, by Theorem 1, we have

$$\alpha(V_h(t)) = \alpha\left(\left\{\frac{1}{2h}\int_{t-h}^{t+h} x(s)ds : x \in V\right\}\right) \leq \frac{1}{h}\int_{t-h}^{t+h} \alpha(V(s))ds,$$

i.e.

$$\alpha(V_h(t)) \leq 2\nu_h(t) \quad \text{for } t \in D, \tag{6}$$

where $\nu_h(t) = \frac{1}{2h}\int_{t-h}^{t+h} \nu(s)ds$. As $\lim_{h\to 0}\|\nu - \nu_h\|_p = 0$, from (4) – (6) we conclude that $\alpha_p(V) \leq 2\left(\int_D \nu^p(t)dt\right)^{1/p}$.

3. Urysohn integral equations. Let F be the mapping defined by

$$F(x)(t) = \int_D f(t,s,x(s))ds \quad (x \in L^p(D,E),\, t \in D).$$

It is known that under the assumptions $1° - 3°$ F continuously maps $L^p(D,E)$ into itself. Assume, in addition, that for any $\varrho > 0$

4° $\lim_{h\to 0}\sup_{\|x\|_p \leq \varrho}\int_D \|F(x)(t+h) - F(x)(t)\|dt = 0$.

Let $(t,s) \to H(t,s)$ be a function from D^2 into R_+ such that $H(t,\cdot) \in L^m(D,R)(m = p/(p-1))$ for a.e. $t \in D$, and the function $t \to \|H(t,\cdot)\|_m$ belongs to $L^p(D,R)$.

THEOREM 3. *If f and g satisfy $1° - 4°$ and*

$$\alpha(f(t,s,Z)) \leq H(t,s)\alpha(Z) \tag{7}$$

for $t, s \in D$ and for each bounded subset Z of E, then there exists a positive number μ such that for any $\lambda \in R$ with $|\lambda| < \eta$ the equation (1) has a solution $x \in L^p(D,E)$.

Proof. Put

$$\eta = \min\left(\sup_{r>0}\frac{r - \|g\|_p}{\|k\|_p(\|a\|_q + br^{p/q})}, \frac{1}{2\|\psi\|_p}\right),$$

where $\psi(t) = \|H(t,\cdot)\|_m$ for $t \in D$. Fix $\lambda \in R$ with $|\lambda| < \eta$. Then there exists $\varrho > 0$ such that

$$\|g\|_p + |\lambda|\|k\|_p(\|a\|_q + b\varrho^{p/q}) \leq \varrho.$$

Let $B = \{x \in L^p(D,E) : \|x\|_p \leq \varrho\}$ and $G(x) = g + \lambda F(x)$ for $x \in B$. Then G is a continuous mapping $B \to B$ and

$$\|G(x)(t)\| \leq \mu(t) \quad \text{for} \quad x \in B \text{ and } t \in D, \tag{8}$$

where $\mu(t) = \|g(t)\| + |\lambda| k(t)(\|a\|_q + b\varrho^{p/q})$. Let V be a countable subset of B such that

$$V \subset \overline{\operatorname{conv}}(G(V) \cup \{0\}). \tag{9}$$

Then

$$V(t) \subset \overline{\operatorname{conv}\,(G(V))(t) \cup \{0\})} \quad \text{for a.e. } t \in D,$$

so that

$$\alpha(V(t)) \leq \alpha(G(V)(t)) \quad \text{for a.e. } t \in D. \tag{10}$$

Put $\nu(t) = \alpha(V(t))$ for $t \in D$. From 4° and (9) it is clear that

$$\lim_{h \to 0} \sup_{x \in V} \int_D \|x(t+h) - x(t)\| dt = 0.$$

Since $\|x(t)\| \leq \mu(t)$ for all $x \in V$ and a.e. $t \in D$, and $\mu \in L^p(D, R)$, this implies

$$\lim_{h \to 0} \sup_{x \in V} \int_D \|x(t+h) - x(t)\|^p dt = 0.$$

Hence, by Theorem 2, $\nu \in L^p(D, R)$ and

$$\alpha_p(V) \leq 2\|\nu\|_p. \tag{11}$$

Fix now $t \in D$ such that $k(t) < \infty$. From 3° and (8) it follows that

$$\|f(t, s, x(s))\| \leq \eta(s) \quad \text{for } x \in V \text{ and a.e. } s \in D,$$

where the function $s \to \eta(s) = K(t,s)(a(s) + b\mu^{p/q}(s))$ belongs to $L^1(D, R)$. Thus, by (7), (10) and Theorem 1, we get

$$\begin{aligned} \alpha(V(t)) &\leq |\lambda| \alpha \left(\left\{ \int_D f(t, s, x(s)) ds : x \in V \right\} \right) \\ &\leq 2|\lambda| \int_D \alpha(\{f(t, s, x(s)) : x \in V\}) ds \leq 2|\lambda| \int_D H(t,s) \alpha(V(s)) ds, \end{aligned}$$

i.e. $\nu(t) \leq 2|\lambda| \int_D H(t,s)\nu(s) ds$. Consequently, by the Hölder inequality, we have $\nu(t) \leq 2|\lambda|\psi(t)\|\nu\|_p$. As the above inequality holds for a.e. $t \in D$, we obtain $\|\nu\|_p \leq 2|\lambda| \|\psi\|_p \|\nu\|_p$. Since $2|\lambda| \|\psi\|_p < 1$, from this we deduce that $\|\nu\|_p = 0$. Therefore, by (11), $\alpha_p(V) = 0$, i.e. V is relatively compact in $L^p(D, E)$. Applying now Mönch's generalizaton of the Schauder fixed point theorem [4] we conclude that there exists $u \in B$ such that $u = G(u)$. It is clear that u is a solution of (1).

4. Volterra integral equations. Consider now the equation (2). Choose a positive number $a < \min(d, \omega_+)$, where $[0, \omega_+)$ is the maximal interval of existence of the maximal absolutely continuous solution z of the initial value problem

$$z' = (\|g(t)\| + k(t)\|a\|_q + bk(t)z^{1/q})^p, \ z(0) = 0.$$

Let $J = [0, a]$, $\varrho = \max_{t\in J} z(t)$ and $B = \{x \in L^p(J, E) : \|x\|_p \leq \varrho\}$. Put

$$F(x)(t) = \int_0^t f(t, s, x(s))ds \quad (x \in L^p(J, E), t \in J).$$

Assume that $(t, s, u) \to h(t, s, u)$ is a nonnegative function defined for $0 \leq s \leq t \leq d$, $u \geq 0$, satisfying the following conditions:

(i) for any nonnegative $u \in L^p(D, R)$ there exists the integral

$$\int_0^t h(t, s, u(s))ds \quad \text{for a.e.} \quad t \in D;$$

(ii) for any c, $0 < c \leq d$, $u = 0$ a.e. is the only nonnegative function on $[0, c]$ which belongs to $L^p([0, c], R)$ and satisfies

$$u(t) \leq 2\int_0^t h(t, s, u(s))ds \quad \text{almost everywhere on } [0, c].$$

Theorem 4. *If* $1^\circ - 4^\circ$ *hold and*

$$\alpha(f(t, s, Z)) \leq h(t, s, \alpha(Z)) \tag{12}$$

for $t, s \in D$ *and for each bounded subset* Z *of* E*, then the equation (2) has at least one solution* $x \in L^p(J, E)$.

Proof. Put $G(x) = g + F(x)$ for $x \in B$. Then G is a continuous mapping of B into $L^p(J, E)$. For any positive integer n we define a function u_n by

$$u_n(t) = \begin{cases} g(t) & \text{if } 0 \leq t \leq a/n \\ g(t) + \int_0^{t-a/n} f(t, s, u_n(s))ds & \text{if } a/n \leq t \leq a. \end{cases}$$

By the Hölder inequality from 3° it follows that

$$\|u_n(t)\| \leq \|g(t)\| + k(t)\|a\|_q + bk(t)\left(\int_0^t \|u_n(s)\|^p ds\right)^{1/q} \tag{13}$$

and

$$\|u_n(t) - g(t) - \int_0^t f(t, s, u_n(s))ds\| \leq k_n(t)(\|a\|_q + b(\int_0^t \|u_n(s)\|^p ds)^{1/q}) \tag{14}$$

for $t \in J$, where

$$k_n(t) = \begin{cases} k(t) & \text{if } 0 \leq t \leq a/n \\ \|K(t, \cdot)_{\chi[t-a/n,t]}\|_r & \text{if } a/n \leq t \leq a. \end{cases}$$

Putting $z_n(t) = \int_0^t \|u_n(s)\|^p ds$, we see that $z_n(0)$ and

$$z_n'(t) \leq (\|g(t)\| + k(t)\|a\|_q + bk(t)z_n^{1/q}(t))^p \quad \text{for a.e. } t \in J.$$

Hence, applying the theorem on differential inequalities, we conclude that $z_n(t) \le z(t) \le \varrho$ for $t \in J$. Consequently, $u_n \in B$ and, by (13),

$$\|u_n(t)\| \le \mu(t) \quad \text{for } t \in J,$$

where $\mu(t) = \|g(t)\| + k(t)(\|a\|_q + b\varrho^{p/q})$. On the other hand, as $\lim_{n\to\infty} k_n(t) = 0$ and $k_n(t) \le k(t)$ for a.e. $t \in J$, (14) implies that

$$\lim_{n\to\infty}(u_n(t) - G(u_n)(t)) = 0 \quad \text{for a.e. } t \in J$$

and

$$\lim_{n\to\infty}\|u_n - G(u_n)\|_p = 0. \tag{15}$$

Arguing similarly as in the proof of Theorem 3, we can show that the set $\{u_n : n = 1, 2, \dots\}$ is relatively compact in $L^p(J, E)$. Thus we can find a subsequence (u_{nj}) of (u_n) which converges in $L^p(J, E)$ to a limit u. In view of (15), this implies that $u = G(u)$, so that u is a solution of (2).

5. Volterra-Hammerstein integral equations. In this section we shall apply the result from Section 4 to the equation (3). Assume that

5° $(s, x) \to \varphi(s, x)$ is a function from $D \times E$ into a Banach space M which is strongly measurable in s and continuous in x, and

$$\|\varphi(s, x)\| \le a(s) + b\|x\|^{p/q} \quad \text{for } s \in D \text{ and } x \in E;$$

6° Q is a strongly measurable function from D^2 into the space of continuous linear mappings $M \to E$ such that $\|Q(t, s)\| \le K(t, s)$ for $t, s \in D$. Furthermore, we assume that

7° $p \ge g$ and $1 \le l \le p/q$; let i be such that $1/r + 1/i + l/p = 1$ and $1 < i \le \infty$; or

7°′ $p \ge 2$, $1 \le l \le p - 1$ and $K \in L^p(D^2, R)$; let i be such that $1 < i \le \infty$ and $1/p + 1/i + l/p = 1$.

THEOREM 5. *If g, φ and Q satisfy 1°, 5°, 6°, and 7° or 7°′, and there exists a function $c \in L^i(D, R)$ such that $\alpha(\varphi(t, Z)) \le c(t)\alpha^l(Z)$ for a.e. $t \in D$ and for each bounded subset Z of E, then the equation (3) has a solution $x \in L^p(J, E)$.*

Proof. Putting $f(t, s, x) = Q(t, s)\varphi(s, x)$ we observe that from 5° and 6° it follows that the corresponding operator F satisfies 4°. Moreover, f satisfies (12) with $h(t, s, u) = \|Q(t, s)\|c(s)u^l$. Hence Theorem 5 is an immediate consequence of Theorem 4.

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A. Mickiewicz University
Poznan, Poland

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