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A FUNCTIONAL APPROACH TO THE THEORY OF PRIME IMPLICANTS

Frank Markham Brown and **Sergiu Rudeanu**

Abstract. The theory of prime implicants has been developed independently to simplify truth-functions (Quine, 1952) and to solve inferential problems in propositional logic (Blake, 1937). The object of this paper is to generalize Blake's approach, which unlike Quine's is little known, in the setting of function theory. We begin by developing an axiomatic theory of prime implicants within the general framework of finite join semilattices; Blake's concepts of syllogistic representation and canonical form are defined naturally within this framework. We next specialize this axiomatic theory to simple Boolean functions (equivalently, propositional functions) to obtain the classical theory of prime implicants. Finally, we derive the theory of prime implicants for general Boolean functions, together with a few results specific to such functions.

A. Introduction. Although the algebra of logic was invented by George Boole as a medium for deduction, systematic methods of interference are seldom applied to Boolean (i.e., propositional) problems. Such methods would appear to have many applications, however, in switching theory, logical design, fault-diagnosis, and other fields in which Boolean algebra is used.

The resolution principle, formulated by Robinson [7] in 1965, enables deduction in the predicate calculus to be mechanized by means of a single rule of inference. This principle is of course applicable to Boolean problems, inafmuch as propositional logic is a subset of predicate logic. The basic approach to resolution applied in predicate calculus – theorem-proving by refutation – has nevertitheless found little application in switching theory and related fields. An approach to propositional resolution given by Archie Blake [1] seems to us, however, to be applicable in a direct way to such problems.

Blake's dissertation, published 27 years before Robinson's paper, demonstrated that all of the consequents of a disjunctive normal form may be generated by repeated production of the consensus (propositional resolution) of pairs of terms, and that all of the prime implicants of the original function will be included in the resulting formula. Blake's dissertation is remarkable not only for presenting the

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essential idea of resolution, but also for anticipating many of the techniques later discovered by Quine [4,5,6] and others for generating prime implicants.

The objective of the large body of research inspired by Quine's work on prime implicants has been to simplify propositional formulas; Blake, on the other hand, sought a simple characterization of all of the consequents of the Boolean equation $f = 0$, i.e., all Boolean equations $g = 0$ such that $f = 0 \Rightarrow g = 0$. This is a central problem in classical logic, since a collection of simultaneously-asserted propositions may be expressed equivalently by a single Boolean equation of the form $f = 0$.

The theory of prime implication has thus arisen independently to serve two quite different ends, viz., formula-minimization (Quine) and propositional inference via resolution (Blake). While Quine's approach has been the basis for extended research, Blake's formulation remains virtually unknown.

Blake noted that the problem of finding consequents $g = 0$ of the equation $f = 0$ is essentially that of finding functions g such that $g \leq f$. This problem is solved in Blake's formulation by expressing the function f in a form he called "syllogistic"; this form enables all included disjunctive normal expressions to be read off by inspection.

Among the syllogistic forms for a Boolean function f is one which Blake called the "simplified canonical form", and which we shall call *the Blake canonical form* for f and denote by BCF(f). This form turns out to the disjunction of all of the prime implicants of f.

Our object in this and future work is to clarify and extend Blake's approach to propositional inference. In the present paper we focus on Blake's theory of prime implicants, formulating the essentials of that theory from an entirely function-theoretical point of view, and presenting some applications arising from that formulation. We show in the process that Boolean expressions (disjunctive normal formulas) are not essential to the theory of prime implicants.

We begin by developing an axiomatic theory of prime implicants within the general framework of finite join semilattices; the concepts of syllogistic representation and Blake canonical form are defined naturally within this framework. We next specialize this axiomatic theory to simple Boolean functions (equivalently, propositional functions) to obtain at once the classical theory of prime implicants. Finally, we derive the theory of prime implicants for the most general Boolean functions, together with a few results specific to such functions.

B. Axiomatic Formulation. In a disjunctive representation of a Boolean function, say

$$f = t_1 + \cdots + t_m, \tag{1}$$

where $t_1, \ldots, t_m$ are terms, we customarily apply associativity, commutativity and idempotence, so that we may disregard parentheses, order of terms and possible repetition of terms: two disjunctive representations that differ only in these three respects are thus identified. We therefore define a disjunctive representation of a function f as a set $\{t_1, \ldots, t_m\}$ of terms such that identity (1) holds. The elements of the set $\{t_1, \dots, t_m\}$ are functions, no matter how they are represented, while in a disjunctive form each term t is to be represented in the canonical form. This dif-

$$t = x_{i_1}^{e_{i_1}} \cdots x_{i_k}^{e_{i_k}}. \tag{2}$$

ference is immaterial, however, and (as we show) the essential of the entire theory of prime implicants can be recaptured within the function-theoretical framework.

The function-theoretical aspect is convenient, moreover, for generalizations. Consider, for example, the Boolean functions dealt with in the conventional theory. Such functions involve variables, but not constants; they are what we have called simple Boolean functions [8] (or, equivalently, propositional functions). The difficulty in obtaining a theory of prime implicants for arbitrary Boolean functions, whose representation may involve constants as well as variables, lies in the construction of a convenient concept of formal expression. This difficulty is avoided in the functional approach. In addition, the functional theory can be presented in the more general setting of lattices and semilattices, as suggested by Davio, Deschamps and Thayse [2].

Let (L, V) be a finite join semilattice. By a *V-generating system* (or simply *generating system*) we mean a subset G of L such that every element of L can be written as a join of elements from G.

Note that a generating system, e.g., L itself, always exists in the semilattice. We work in the sequel with an arbitrary but fixed generating system G.

Remark 1. If L has a least element 0 (zero), then $0 \in G$.

By an *implicant* of an element $a \in L$ we mean an element $g \in G$ such that $g \leq a$; the maximal elements of the set of all implicants of a will be called *prime implicants* of a.

Remark 2. For every implicant g of a there is a prime implicant p of a such that $g \leq p$.

By a *representation* of an element $a \in L$ we mean a subset $\{a_1, \dots, a_s\}$ of L such that

$$a = a_1 \vee \cdots \vee a_s; \tag{3}$$

for the sake of simplicity, however, we shall refer to (3) itself as a representation of a. We nevertheless keep in mind the exact definition given above for a representation, so that we will idetify every pair of representations (3) and $a = a'_1 \vee \cdots \vee a'_t$ which differ only in the order of terms and or the number of occurrences of each term.

A representation (3) with $a_1, \dots, a_s \in G$ will be called a *G-representation*.

PROPOSITION 1. [2, Theorem 1.15]. *In a finite join semilattice, every element equals the disjunction of all its prime implicants.*

Proof. Given $a \in L$ let $\{p_1, \dots, p_m\}$ be the set of all prime implicants of a. Then $p_i \leq a$ $(i = 1, \dots, m)$ and therefore $p_1 \vee \cdots \vee p_m \leq a$. To prove the converse inequality, let $a = g_1 \vee \cdots \vee g_n$ be a G-representation of a. In view of Remark 2, for every integer $h \in \{1, \ldots, n\}$ there is a prime implicant p_{i_h} of a such that $g_h \leq p_{i_h}$. Then

$$a = g_1 \vee \cdots \vee g_n \leq p_{i_1} \vee \cdots \vee p_{i_n} \leq p_1 \vee \cdots \vee p_m \leq a,$$

and therefore

$$a = p_1 \vee \cdots \vee p_m. \tag{4}$$

We shall refer to the representation (4) as the Blake canonical form of a, and write it as BCF(a). It is clear that BCF(a) is unique.

An important property of the Blake canonical form is related to the following definition. Let (3) and

$$b = b_1 \vee \cdots \vee b_t \tag{5}$$

be two G-representations. Following Blake, we say that (5) is formally included in (3), written (5) $\ll$ (3) or $\{b_1, \ldots, b_t\} \ll \{a_1, \ldots, a_s\}$ or $b_1 \vee \cdots \vee b_t \ll a_1 \vee \cdots \vee a_s$, if for every b_i there is some a_h such that $b_i \leq a_h$. It is clear that

$$b_1 \vee \cdots \vee b_t \ll a_1 \vee \cdots \vee a_s \Rightarrow b_1 \vee \cdots \vee b_t \leq a_1 \vee \cdots \vee a_s.$$

A G-representation (3) will be called *syllogistic* in case

$$b_1 \vee \cdots \vee b_t \leq a_1 \vee \cdots \vee a_s \Rightarrow b_1 \vee \cdots \vee b_t \ll a_1 \vee \cdots \vee a_t.$$

for every subset $\{b_1, \ldots, b_t\} \subseteq G$.

Remark 3. The G-representation (3) is *syllogistic* if and only if

$$g \leq a_1 \vee \cdots \vee a_s \Rightarrow g \ll a_1 \vee \cdots \vee a_s$$

for every element $g \in G$.

PROPOSITION 2. *In a finite join semilattice, the G-representation* (3) *is syllogistic if and only if it contains all the prime implicants of a.*

Proof. Let $\{p_1, \ldots, p_m\}$ be the set of prime implicants of a. Suppose $\{p_1, \ldots, p_m\} \subseteq \{a_1, \ldots, a_s\}$. If g is an implicant of a, then $g \leq p_i$ for some $i \in \{1, \ldots, m\}$. But $p_i \in \{a_1, \ldots, a_s\}$; hence $g \ll a_1 \vee \cdots \vee a_s$ and therefore (3) is syllogistic by Remark 3. Conversely, suppose there is some prime implicant p_i of a such that $p_i \notin \{a_1, \ldots, a_s\}$. Then (3) is not syllogistic, for (i) $p_i \leq a_1 \vee \cdots \vee a_s$ but (ii) not $p_i \ll a_1 \vee \cdots \vee a_s$. To show ($ii$), suppose $p_i \leq a_h$ for some $h \in \{1, \ldots, s\}$; then the maximality of p_i implies that $p_i = a_h$, a contradiction.

COROLLARY. *The Blake canonical form is syllogistic.*

The introduction of certain supplementary hypotheses enables us to obtain further results.

LEMMA 1. [2, Theorem 1.16]. *Suppose $(L, \vee, \wedge)$ is a finite lattice and let $a_1, \ldots, a_k \in L$. Then every prime implicant of $a_1 \wedge \cdots \wedge a_k$ is a prime implicant of $p_{h_1} \wedge \cdots \wedge p_{h_k}$ for some prime implicants p_{h_j} of a_j $(j = 1, \ldots, k)$.*

Proof. Let p be a prime implicant of $a_1 \wedge \cdots \wedge a_k$. Then for every $j \in \{1, \ldots, k\}$ we have $p \leq a_j$: hence $p \leq p_{h_j}$ for some prime implicant p_{h_j} of a. It follows that $p \leq p_{h_1} \wedge \cdots \wedge p_{h_k}$ and if $g \in G$ fulfills $p \leq g \leq p_{h_1} \wedge \cdots \wedge p_{h_k}$ then $p \leq g \leq a_1 \wedge \cdots \wedge a_k$; thus $p = g$ by the maximality of p. Therefore p is a prime implicant of $p_{h_1} \wedge \cdots \wedge p_{h_k}$.

Proposition 3. *Suppose $(L, \vee, \wedge)$ is a finite lattice and the meet of every pair of elements of G is in G. For any elements $a_1, \ldots, a_k \in L$, every prime implicant of $a_1 \wedge \cdots \wedge a_h$ is of the form $p_{h_1} \wedge \cdots \wedge p_{h_k}$ for some prime implicants p_{h_j} of a_j $(j = 1, \ldots, k)$.*

Proof. Lemma 1 yields $p \leq p_{h_1} \wedge \cdots \wedge p_{h_k} \leq a_1 \wedge \cdots \wedge a_k$. By hypothesis, however, $p_{h_1} \wedge \cdots \wedge p_{h_k} \in G$; therefore, by the maximality of p, $p = p_{h_1} \wedge \cdots \wedge p_{h_k}$.

Suppose further that L is finite distributive lattice. For each $k \geq 2$ and for every system

$$a_j = a_{j1} \vee a_{j2} \vee \cdots \vee a_{jn(j)} \qquad (j = 1, \ldots, k), \tag{6.j}$$

of k representations, define (6.1) $\times$ (6.2) $\times$ ... $\times$ (6.k) to be the representation of $a_1 \wedge \cdots \wedge a_k$ obtained by multiplying out the k representations (6.1), ..., (6.k). In the other words,

$$a_1 \wedge \cdots \wedge a_k = \vee_{\varphi} a_{1\varphi(1)} \wedge \cdots \wedge a_{k\varphi(k)}, \qquad (6.1) \times \ldots \times (6.k)$$

where φ runs over the set of all functions

$$\varphi : \{1, \ldots, k\} \to \cup_{j=1}^{k} \{1, \ldots, n(j)\} \tag{7}$$

having the property

$$\varphi(j) \in \{1, \ldots, n(j)\} \qquad (j = 1, \ldots, k). \tag{8}$$

Proposition 4. *Suppose $(L, \vee, \wedge)$ is a finite distributive lattice and the meet of every pair of elements of G is in G. For every $k \geq 2$ and every $a_1, \ldots, a_k \in L$, if (6.1), ..., (6.k) are syllogistic G-representations of $a_1, \ldots, a_k$, respectively, then (6.1) $\times \ldots \times$ (6.k) is a syllogistic G-representation of $a_1 \wedge \cdots \wedge a_k$.*

Proof. In view of Proposition 2, it suffices to prove that every prime implicant of $a_1 \wedge \cdots \wedge a_k$ is a term of the representation (6.1) $\times \cdots \times$ (6.k). According to Proposition 3, p is of the form $p = p_{h_1} \wedge \cdots \wedge p_{h_k}$, for some prime implicants p_{h_j} of a_j $(j = 1, \ldots, k)$. But each p_{h_j} belongs to the syllogistic representation (6.j) $j = (1, \ldots, k)$; therefore p does occur in the representation (6.1) $\times \cdots \times$ (6.k).

Corollary. *$BCF(a_1) \times \cdots \times BCF(a_k)$ is a syllogistic representation of $a_1 \wedge \cdots \wedge a_k$.*

C. Specialization to the Classical Theory. The classical theory of prime implicants refers to truth functions, i.e., functions with arguments and values in the two-element Boolean algebra. However, in view of the isomorphism between truth functions and simple Boolean functions over an arbitrary Boolean algebra [8, Corollary to Theorem 1.21], the theory applies exactly as it stands to the latter functions. We wish now to obtain this theory as a particular case of the axiomatics described above; then, for the sake of completeness and in view of applications to be described in subsequent work, we prove a few well-known results specific to prime implicants of simple Boolean functions.

Let $(B, +, \cdot, ', 0, 1)$ be a Boolean algebra and n a positive integer, both of them arbitrary but fixed in the sequel. We recall that a map $f : B^n \to B$ is termed a *simple Boolean function* if it can be obtained from variables by super-positions of the basic operations $+, \cdot, '$.

We take as lattice $(L, +, \cdot)$ the set of 2^{2^n} simple Boolean functions $f : B^n \to B$, endowed with the operations $+, \cdot$ and partial order $\leq$ defined pointwise from the corresponding operations $+, \cdot$ and partial order $\leq$ of B. The set of terms consists of the constant function 1 and all the functions $t : B^n \to B$ that can be written in the form

$$t(x_1, \dots, x_n) = x_{i_1}^{e_{i1}} \cdots x_{i_k}^{e_{ik}} \qquad x_1, \dots, x_n \in B,$$

where the indices $i_1, \dots, i_k$ are distinct members of $\{1, \dots, n\}$, $e_{i_1}, \dots, e_{i_k} \in \{0, 1\}$, and x^e means x or x' according as $e = 1$ or $e = 0$. Note that the expression (9) is unique up to the order of enumeration of indices. The generating system G consists of all the terms plus the constatant function 0 (cf. Remark 1).

To establish the junction with the classical theory, let us first mention:

PROPOSITION 5. *Let* $t = x_{h_1}^{e_{h_1}} \cdots x_{h_j}^{e_{h_j}}$ *and* $\nu = x_{i_1}^{d_{i_1}} \cdots x_{i_k}^{d_{i_k}}$ *be representations of the terms* t *and* ν *in the form (9) described above. Then* $t \leq \nu$ *if and only if* $\{i_1, \dots, i_k\} \subseteq \{h_1, \dots, h_j\}$ *and* $e_{i_r} = d_{i_r}$ $(r = 1, \dots, k)$.

In other words, $t \leq \nu$ if and only if every *literal* $x_{i_r}^{d_{i_r}}$ of ν iv in t. ir

Proof. Well-known.

We see therefore that the operations $+, \cdot$ and the order $\leq$ dealt with in the classical theory of Boolean prime implicants coincide with those obtained by specializing the general concepts of the axiomatic approach described in Section B. As we have remarked in the introduction, the classical concept of *disjunctive form* is slightly stronger than the specialization of the concept of G-representation: the terms have to be written in the standard form (9) and the constant function $0 \in G$ is always omitted from a disjunctive form of a function $f \neq 0$. The results of the axiomatic theory which involve the concept of G-representation however, are Remark 3, Propositions 2,4 and their corollaries, which refer to syllogistic representations and Blake cannonical forms (these two concepts disregard the "internal representation" of G-elements). Thus each of the properties mentioned above remains valid when G-representations are replaced by disjunctive forms, in view of the following scheme: the (syllogistic, Blake) disjunctive forms given in the hypothesis are (syllogistic, Blake) G-representations, and from the (syllogistic, Blake) G-representations obtained in the conclusion we obtain immediately (syllogistic, Blake) disjunctive forms, as desired.

The next two propositions are specific to the theory of prime implicants of simple Boolean functions and make use of the following concept. Two terms s and t are said to have an opposition provided there is a variable x_i (more rigorously, a projection-function) such that either (i) $s \leq x_i'$ and $t \leq x_i'$ or (ii) $s \leq x_i'$ and $t \leq x_i$.

This happens if and only if the letter x_i appears uncomplemented in the standard form (9) of one of the terms and complemented in the corresponding form of the other term. Two terms may have several oppositions, a single opposition, or none.

PROPOSITION 6. [1, Corollary to Theorem 10.3]. *Let r and s be terms having exactly one opposition, say $r = xr_1$ and $s = x's_1$ where the terms r_1 and s_1 are independent of x and have no opposition. Then*

$$r + s + r_1 s_1 \tag{10}$$

is a syllogistic representation of $r + s$.

Proof. Take $a = b = 1$ in Proposition 10 below.

PROPOSITION 7. *Let f be exprersed by*

$$f = t_1 + \cdots + t_m, \tag{1}$$

where $t_1, \ldots, t_m$ are terms such that for every pair t_{j_1}, t_{j_2} there is no opposition. Then (1) is syllogistic.

Proof. Let $p = x_{h_1}^{e_{h_1}} \cdots p_{h_k}^{e_{h_k}}$ be a prime implicant of f. Suppose first that every term t_j contains a literal $x_{i(j)}^{e_{i(f)}}$ not in p. Define $x_{h_r}^* = e_{h_r}$ $(r = 1, \cdots, k)$, $x_{i(j)}^* = e_{i(j)} \oplus 1$ $(j = 1, \ldots, m)$ (where $x \oplus y = x'y + xy'$), which is possible because $\{h_1, \ldots, h_k\} \cap \{i(1), \ldots, i(m)\} = \emptyset$ and $i(j_1) = i(j_2) \Rightarrow e_{i(j_1)} = e_{i(j_2)}$ for all j_1, $j_2 \in \{1, \ldots, m\}$. Take $x^* \in B^n$ defined as above in the components, if any. Then $p(x^*) = 1$ and evry $t_j(x^*) = 0$ $(j = 1, \ldots, m)$, which contradicts $p \leq f$. Hence there exists a term t_{j_0} such that all its literals are in p, and therefore $p \leq t_{j_0}$ by Proposition 5; but $t_{j_0} \leq f$, and therefore $p = t_{j_0}$ by the minimality of p.

We conclude this section with

Remark 4. ([Kuntzmann [3, IV. 13]). The converse of Proposition 3 is not valid, as shown by the following example. Let $BCF(f_1) = xy + xz + yz'$ and $BCF(f_2) = xy + xz' + yz$. Then $BCF(f_1 f_2) = BCF(xy + xyz + xyz') = xy$, while, for example, $(xy)(yz) = xyz$.

D. Extension to Arbitrary Boolean Functions. We seek now to extend the theory of prime implicants from simple Boolean functions to arbitrary Boolean functions. We recall that a map $f : B^n \to B$ is said to be a *Boolean function* if it can be obtained from constants and variables by superposition of the basic operations $+$, $\cdot$, $'$. An equivalent definition is that Boolean functions are those functions that can be obtained from simple Boolean functions by fixing the values of some (possibly none) of their variables. See [8] for a detailed study of the distinction between Boolean functions and simple Boolean functions.

The Boolean algebra B will henceforth be supposed to be finite. This hypothesis is not overly restrictive, because in practice we work in most cases with a finite number of Boolean functions, each of which is expressed using a finite number of constants. We can therefore replace the original Boolean algebra by the Boolean subalgebra generated by all these (finitely many) constants; hence the latter subalgebra is finite.

To apply the axiomatic theory, we take as lattice $(L, +, \cdot)$ the set of Boolean functions $f : B^n \to B$, endowed with the operations $+$, $\cdot$ and the partial order $\leq$ defined pointwise from the corresponding operations $+$, $\cdot$ and partial order $\leq$ of B. We take as generating system G the set of all *generalized terms* or *genterms*, i.e., the set of all functions that can be written in the form at, where $a \in B$ and $t : B^n \to B$ is a term in the previous sense of definition (9). Clearly $(L, +, \cdot)$ is a finite distributive lattice, the genterms form a generating system, and the meet of two genterms is always a genterm. The axiomatic theory therefore applies to this case.

In the remainder of this paper we give a necessary condition for a genterm to be a prime implicant of a Boolean function (Proposition 8) and prove that our concept of prime implicant reduces to the customry one in the case of a simple Boolean function (Proposition 9). We also generalize Proposition 6 to the case of arbitrary Boolean functions (Proposition 10) and show that such generalization is not possible for Proposition 7; cf. Remark 5.

LEMMA 5. *For all genterms as, bt,*

$$as \leq bt \Leftrightarrow [a \leq b \text{ and } s \leq t] \quad \text{or} \quad [a = 0]. \tag{11}$$

Proof. Suppose $as \leq bt$ and $a \neq 0$. Then $as \leq b$ and, taking $x \in B^n$ such that $s(x) = 1$, we obtain $a \leq b$. Now suppose, by way of contradiction, that not $s \leq t$. Then $s(x^*) = 1$ and $t(x^*) = 0$ for some $x^* \in B^n$, which contradicts $as \leq bt$. The converse is trivial.

COROLLARY. *For all genterms as, bit,*

$$as = bt \Leftrightarrow [a = b \text{ and } s \leq t] \quad \text{or} \quad [a = 0]. \tag{12}$$

PROPOSITION 8. *If cp is a prime implicant of a Boolean function, where $c \in B$ and p is a term, then*

$$c = \prod_i [f(i) + p'(i)] \in f(B^n), \tag{13}$$

where i runs over $\{0,1\}^n$.

Proof. According to a theorem of Whitehead [9] (see also [8, Theorem 2.5]),

$$\{f(x) \mid x \in p^{-1}(1)\} = [a, b] = \{y \in B \mid a \leq y \leq b\} \tag{14}$$

where

$$a = \prod_i [f(i) + p'(i)] \text{ and } b = \sum_i f(i)p(i). \tag{15}$$

Thus $[a, b] \subseteq f(B^n)$ and we will prove that $c = a$. Note that ap is an implicant of f, because if $x \in p^{-1}(0)$ then $ap(x) = 0$, while if $x \in p^{-1}(1)$ then $ap(x) = a \leq f(x)$. Now take $x^* \in p^{-1}(1)$ such that $f(x^*) = a$. Thus $c \leq a$, and in fact $c = a$; otherwise $c < a$ and therefore $cp(x) < ap(x)$ for $x \in p^{-1}(1)$ and, since $cp \leq ap$, it follows that $cp \leq ap$, which contradicts the maximality of cp.

PROPOSITION 9. *Let f be a simple non-zero Boolean function. Then the prime implicants of f obtained within the genterms are the same as the prime implicants of f in the customary sense.*

Proof. Let p be a prime implicant of f in the customary sense. In view of Remark 2, $p \leq cq$ for some genterm cq that is a prime implicant of f. But Lemma 2 implies that $c = 1$ and $p \leq q$; hence $p = q$ by the maximality of p within term-implicants. Thus $p = cq$ is a prime implicant in the sense of genterms. Conversely, let cp be a prime implicant of f, where c fulfills condition (13) in Proposition 8. It follows from [8], Theorem 1.7, that $f(i)$, $p(i) \in \{0,1\}$ for all $i \in \{0,1\}^n$; hence $c \in \{0,1\}$ by (13). But $c \neq 0$ because 0 is not a prime implicant of the nonzero function f. Hence $c = 1$, and therefore $cp = p$ is a prime implicant in the conventional sense.

PROPOSITION 10. *Let r and s be genterms having exactly one opposition, say $r = axr_1$ and $s = bx's_1$, where a, $b \in B$ and r_1, s_1 are term independent of the variable x and having no opposition. Then*

$$r + s = r + s + abr_1s_1 \tag{16}$$

is a syllogistic representation.

Proof. Inafmuch as

$$\begin{aligned} r + s + abr_1s_1 &= axr_1 + bx's_1 + (x + x')abr_1s_1 \\ &= axr_1 + bx's_1 = r + s, \end{aligned}$$

we see that (16) holds. Further, let cp be a primer implicant of $r+s$; we must show that $cp \in \{r, s, abr_1s_1\}$. If $p = xp_1$, where p_1, is a term independent of x, then multiplying

$$cp \leq r + s = axr_1 + bx's_1 \tag{17}$$

by x we obtain $cp = cxp_1 \leq axr_1 = r \leq r + s$; therefore $cp = axr_1 = r$ by the maximality of cp. If $p = x'p_1$, one proves similarly that $cp = s$. If p is independent of x, then taking in turn $x = 1$ and $x = 0$ in (17), we obtain $cp \leq ar_1$ and $cp \leq bs_1$; hence $cp \leq abr_1s_1$. But it follows from (16) that $abr_1s_1 \leq r + s$, and therefore $cp = abr_1s_1$ by the maximality of cp.

Remark 5. Proposition 7 cannot be extended to genterms. Take two elements a, $b \in B \setminus \{0,1\}$ such that $a + b = 1$, for example, and notice that xy is a prime implicant of $f(x,y) = ax + by$.

We conclude with two results that will be useful in extensions of the present work.

LEMMA 3. *If a prime implicant p of a Boolean function $f : B^n \to B$ depends actually on a variable $x_i (\leq i \leq n)$, then so does f.*

Proof. Let $p = ap_1$, where $a \in B$ and p_1 is a term. Then the standard form (9) of p_1 contains the literal $x_i^{e_i}$ with $e_i \in \{0,1\}$. Taking all literals of p_1 equal to 1 makes $f(x) \geq a$ and there is a point $y \in B^n$ for which $y_i^{e_i} = 0$, the other literals of p_1 are equal to 1, and $f(y) \not\geq a$, otherwise dropping $x_i^{e_i}$ from p_1 would result in a new implicant, q, of f with $p < q$, which is a contradiction. Thus f depends actually on x_i.

LEMMA 4. *Let t and ν be genterms with $0 \neq t \leq \nu$. If t is independent of a variable x, then so is ν.*

Proof. Let $t = at_1$, $\nu = b\nu_1$ and write the terms t_1 and ν_1 in the standard form (9). Then $t_1 \leq \nu_1$ by Lemma 2 and it follows from Proposition 5 that ν_1 is independent of x.

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Department of Electrical Engineering
University of Kentucky USA

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Faculty of Mathematics
University of Bucharest
Bucharest, Romania