PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série tome 40 (54), 1986, pp.3–9

# ON THE LARGEST EIGENVALUE OF SOME HOMEOMORPHIC GRAPHS

**Slobodan K. Simić, Vlajko Lj. Kocić**

**Abstract.** Two particular classes of mutually homeomorphic graphs are considered. For any two graphs of the same class, the relationship between the structure and the largest eigenvalue is discussed. Some relevant applications are outlined.

## 0. Introduction

We will consider only undirected graphs without loops or multiple edges. The terminology concerning graphs will follow [1]; for all details on graph spectra, not given here, see [2].

There are many results in the literature concerning the largest eigenvalue of a graph and the graph structure (see [2] for details). In this paper we are mainly interested to get some conclusions relating the largest eigenvalue of a graph and the graph structure, provided the graph is modified locally. Our motivation for this kind of investigations stems from the following problem of D. M. Cvetković (see [4], p. 211):

Let $\rho(k)$ be the largest eigenvalue of the graph obtained from the cycle $C_n$ on $n$ vertices ($n \geq 6$) by adding an edge between two vertices at distance $k$ ($k = 2, 3, \dots, [n/2]$). Prove or disprove that $\rho(k)$ is monotone.

In the next section we will prove that the monotonicity holds even for some larger classes of graphs; some possible applications of our main results are discussed in the last section.

## 1. The main results

Let $A$ be the adjacency matrix of a graph, $\rho$ its largest eigenvalue, and $x$ the corresponding eigenvector. For further reference, we shall rewrite the basic relation $Ax = \rho x$, in the form

$$(1) \qquad \rho x_i = \sum_{r \sim i} x_r,$$

---

*AMS Subject Classification* (1980): Primary 05C50.

so that is refers, in particular, to the $i$-th vertex. In this section we will find a relationship between the largest eigenvalue of a graph and a set of invariants sufficient to determine a graph. The graphs to be examined belong to the following classes.

**Class 1: Paralel paths.** The graphs of this class are all homeomorphic (topologically equivalent according to [6]) to a multigraph consisting of $k$ paralel edges; in fact they are obtained by introducing the vertices of degree two into the edges of the multigraph. By $P(n,k)$ we shall denote the corresponding graphs with $n$ vertices. Any graph from $P(n,k)$ has $m(=n+k-2)$ edges and is determined up to isomorphism, by a $k$-tuple $(m_1, m_2, \ldots, m_k)$, where $m_1 \geq m_2 \geq \cdots \geq m_k$, while $m_i$ is the number of edges in its $i$-th path (paths are ordered according to their lenghts). We first prove the following theorem.

THEOREM 1. *Let $P(m_1, m_2, \ldots, m_k)$ be any graph from $P(n,k)$ while $\rho = \rho(m_1, m_2, \ldots, m_k)$ is its largest eigenvalue. If all $m_r'$ s are fixed except, say $m_i$ and $m_j$, then $\rho$ is an increasing function in $|m_i - m_j|$.*

*Proof.* For convenience, let $G = P(m_1, m_2, \ldots, m_k)$. Next, let

$$x = (u, x_1^1, \ldots, x_{m_1-1}^1, \ldots, x_1^s, \ldots, x_{m_s-1}^s, \ldots, x_1^k, \ldots, x_{m_k-1}^k, \nu)$$

be the corresponding eigenvector of $\rho$ (see Fig. 1).

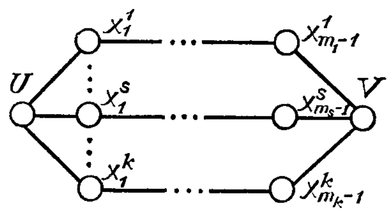


**Fig. 1**

Also, we assume that

$$x_0^1 = x_0^2 = \cdots = x_0^k = u \quad \text{and} \quad x_{m_1}^1 = x_{m_2}^2 = \cdots = x_{m_k}^k = \nu.$$

Note that, since $x$ corresponds to the largest eigenvalue of a connected graph, we have $x > 0$ (0 being 0-vector of the same size). Applying (1) to the interior vertices of the $s$-th path (bold lines in Fig. 1), we get the following difference equations

$$x_{i+2}^s - \rho x_{i+1}^s + x_i^s = 0 \qquad (i = 0, \ldots, m_s - 2), \tag{2}$$

or, if (1) is applied to the exterior (common) vertices of all paths we get

$$\rho u = x_1^1 + \cdots + x_1^k \quad \text{and} \quad \rho\nu = x_{m_1-1}^1 + \cdots + x_{m_k-1}^k. \tag{3}$$

The latter conditions, i.e. (3), may be regarded as boundary conditions for (2). By symmetry, (see [5], p. 166 Lemma 3.3) we also have

$$x_i^s = x_{m_s-i}^s \qquad (i = 1, \ldots, m_s - 1; s = 1, \ldots, k) \tag{4}$$

and in particular, $u = \nu$.

Solving (2) for a fixed $s$, we get

$$x_i^s = a_s r^i + b_s r^{-i}, \tag{5}$$

where $r = 1/2(\rho + \sqrt{\rho^2 - 4}) \quad (r > 1)$.

From (4) it follows that $b_s = a_s r^{m_s}$, and therefore we get

$$x_i^s = a_s(r^i + r^{m_s - i}). \tag{6}$$

Using boundary conditions, from

$$a_1(1 + r^{m_1}) = a_2(1 + r^{m_2}) = \cdots = a_k(1 + r^{m_k})$$

we get

$$a_s = c\Pi_{\substack{t=1\\ t\neq s}}^k (1 + r^{m_t}). \tag{7}$$

Thus we have

$$x_i^s = \Pi_{t=1}^k (1 + r^{m_t}) \frac{r^i + r^{m_s - i}}{1 + r^{m_s}} \qquad (i = 0, \ldots, m_s), \tag{8}$$

if $c$ from (7) is normalized to 1. It can be readily be seen that (8) is valid even if $s = k$ and $m_k = 1$.

Applying (1) to any of the exterior vertices, puting $\rho = 2\mathrm{ch}\, 2t (t > 0)$ or equivalently $r = e^{2t}$, after some usual transformations we get

$$2\mathrm{ch}\,(2t) - \sum_{s=1}^k \frac{\mathrm{ch}\,(m_s - 2)t}{\mathrm{ch}\, m_s t} = 0. \tag{9}$$

Now, we shall establish the behaviour of $\rho = \rho(m_1, \ldots, m_k)$ under the assumptions of the theorem. Since $\rho = 2\mathrm{ch}\, 2t$ $(t > 0)$, we should pay attention to $t$. Consequently, we will examine $t = t(m_1, \ldots, m_i, \ldots, m_j, \ldots, m_k)$ allowing only $m_i$ and $m_j$ to change, while keeping their sum fixed ($= c$, for convenience); if so, $t$ depends only on $m_i$ and $c - m_i$, or, in other words, it is a function of $|m_i - m_j|$. Let

$$F(m_1, \ldots, m_k, t) = 2\mathrm{ch}\,(2t) - \sum_{s=1}^k \frac{\mathrm{ch}\,(m_s - 2)t}{\mathrm{ch}\, m_s t}. \tag{10}$$

Deriving $F$ with respect to $t$ and $m_i$ we get

$$\frac{\delta F}{\delta T} = 4\mathrm{sh}\,(2t) + \sum_{s=1}^k \frac{m_s \mathrm{sh}\, 2t + 2\mathrm{ch}\, m_s t \mathrm{sh}\,(m_s - 2)t}{\mathrm{ch}^{\,2} m_s t} > 0, \tag{11}$$

and

$$\frac{\delta F}{\delta m_i} = t\mathrm{sh}\,(2t) \left( \frac{1}{\mathrm{ch}^{\,2} m_i t} - \frac{1}{\mathrm{ch}^{\,2}(c - m_i)t} \right). \tag{12}$$

Since $\frac{\delta t}{\delta m_i} = -\frac{\delta F/\delta t}{\delta F/\delta m_i}$ we conclude

$$(13) \qquad \delta t/\delta m_i > 0 \ \text{for} \ m_i < m_j, \ \text{while,} \ \delta t/\delta m_i > 0 \ \text{for} \ m_i > m_j.$$

Finally, from (13) we conclude that $\rho$ is an increasing function in $|m_i - m_j|$, completing the proof of the theorem.

*Remark 1.* By considering graphs from $P(m_1, m_2, 1)$ we immediately get the answer to the problem of D. M. Cvetković. According to D. M. Cvetković, P. Rowlinson also solved the same problem by evaluating the characteristic polynomials of the corresponding graphs (see [7]).

Let the $k$-tuple $M' = (m_1', m_2', \dots, m_k')$ be obtained from the $k$-tuple $M = (m_1, m_2, \dots, m_k)$ by deleting $m_1$ and $m_k$ (the largest and the smalest component), and then inserting $\{1/2(m_1 + m_k)\}$ and $[1/2(m_1 + m_k)]$, and reordering the components if necessary. The immediate consequence of the theorem above is:

Corollary 1(a): *If $\rho(G)$ and $\rho(G')$ are the largest eigenvalues of the graphs $G$ and $G'$ from $P(n,k)$ which correspond to the $k$-tuples $M$ and $M'$ respectively, then $\rho(G') < \rho(G)$.*

Applying this corollary repeatedly, we can find the extremal graphs from $P(n,k)$ with respect to the largest eigenvalue, i.e. graphs whose largest eigenvalue attains minimum or maximum value.

Corollary 1(b): *Let $\rho(m_1, m_2, \dots, m_k)$ be the largest eigenvalue of the graph $P(m_1, m_2, \dots, m_k)$ from $P(n,k)$. Then the following holds*

$$\rho(\underbrace{q+1, \dots, q+1}_{r \ \text{times}}, q, \dots, q) \le \rho(m_1, m_2, \dots, m_k) \le \rho(m - 2k + 3, 2, 2, \dots, 2, 1),$$

*where $q = [m/k]$, while $r = m - kq$.*

Using relation (9) we can easily obtain the nex4t result which is already known from [6].

Corollary 1(c): *Let $G$ be any graph consisting of $k$ paralel paths. Then the following holds*

$$k/\sqrt{k-1} < \rho(G) \le (1 + \sqrt{8k-7})/2,$$

*where the lower bound is the best possible (it is in fact a limiting point), while the upper bound is attained with the graph $P(2, 2, \dots, 2, 1)$.*

**Class 2: Cycles with a vertex in common.** These graphs are all homeomorphic to a graph consisting of $k$ loops sharing a common vertex. We now denote by $C(n,k)$ the corresponding graphs with $n$ vertices. Any graph from $C(n,k)$ has $m(= n + k - 1)$ edges and is determined up to isomorphism, by a $k$-tuple $(m_1, m_2, \dots, m_k)$, where $m_1 \ge m_2 \ge \dots \ge m_k$, while $m_i$ is the number of edges in its $i$-th cycle (cycles are ordered according to their lengths).

In what follows, we have a complete analogy with the results from the previous class. Therefore, we will only mention the results.

THEOREM 2. *Let $C(m_1, m_2, \ldots, m_k)$ be any graph form $C(n,k)$, while $\rho = \rho(m_1, m_2, \ldots, m_k)$ is its largest eigenvalue. If all $m_r'$ s are fixed except, say $m_i$ and $m_j$, then $\rho$ is an increasing function in $|m_i - m_j|$.*

It is also worth mentioning that relation (9) now reads:

$$\operatorname{ch}(2t) - \sum_{s=1}^{k} \frac{\operatorname{ch}(m_s - 2)t}{\operatorname{ch} m_s t} = 0. \tag{14}$$

COROLLARY 2(A). *If $\rho(G)$ and $\rho(G')$ are the largest eigenvalues of the graphs $G$ and $G'$ from $C(n,k)$ which correspond to the $k$-tuples $M$ and $M'$ respectively, then $\rho(G') < \rho(G)$.*

COROLLARY 2(B). *For any graph $C(m_1, m_2, \ldots, m_k)$ from $C(n,k)$ we have $\rho(\underbrace{q+1, \ldots, q+1}_{t \text{ times}}, q, \ldots, q) \leq \rho(m_1, m_2, \ldots, m_k) \leq \rho(m - 2k + 3, 2, 2, \ldots, 2, 1)$, where $q = [m/k]$ while $r = m - kq$.*

COROLLARY 2(C). *Let $G$ be any graph consisting of $k$ cycles with a vertex in common. Then the following holds*

$$2k/\sqrt{2k-1} < \rho(G) \leq \sqrt{\sqrt{8k+1}+1}$$

*where the lower bound is the best possible (it is a limiting point), while the upper bound is attained with the graph $C(3, 3, \ldots, 3)$.*

*Remark* 2. Some other classes of graphs could be treated by the same technique. For instance, we can consider paths with a vertex in common, i.e. graphs homeomorphic to a star. This problem was already solved in [8], but in a more general form. In contrast to the results above, in this case the largest eigenvalue is the greatest if all paths are of nearly equal lenght.

## 2. Some applications

In this section we shall outline some applications of the results above. In fact, we shall be concerned with determining, in the set of bicyclic graphs within a fixed number of vertices, those graphs whose largest eigenvalue is the smallest. The analoguous problem with trees and unicyclic graphs is already settled; paths, respectively cycles, of appropriate lengths are the corresponding graphs (see also [3]). With bicyclic graphs the problem is more involved.

For that purpose we shall need the following facts from [6].

An internal path of a graph is a sequence of vertices $x_1, \ldots, x_k$ such that all $x_i$ are distinct (except possibly $x_1 = x_k$), the degrees $d(x_i)$ satisfy $d(x_1) > 3$, $d(x_2) = \cdots = d(x_{k-1}) = 2$ (unless $k = 2$), $d(x_k) > 3$, $x_i$ is adjacent to $x_{i+1}$, $i = 1, \ldots, k-1$.

PROPOSITION 1. *If $xy$ is an edge of a conected graph $G$ not on an internal path, then the largest eigenvalue strictly increases after subdividing the edge $xy$:*

*otherwise, if $xy$ is on an internal path of $G$ while $G$ is nonequal to $T_n$ (see Fig. 2), then the largest eigenvalue strictly decreases after the subdivision.*

Now let $G$ be an arbitrary bicyclic graph with a fixed number of vertices. If it does not contain vertices of degree one, than it looks as one of the graphs of Fig. 3. Otherwise, we can delete any vertex of degree one from $G$. By the interlacing theorem (see [2]), this reduces the largest eigenvalue. If we then insert a new vertex by subdividing any edge belonging to a cycle, andthus to an internal path as well, we will get a bicyclic graph again, but with a smaller number of vertices of degree one. By the proposition above, the latter transformation also reduces the largest eigenvalue. So, on the basis of these observations, we can restrict ourselves only to bicyclic graphs as in Fig. 3.

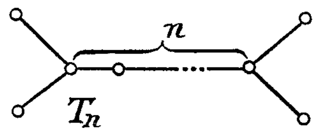


**Fig. 2**

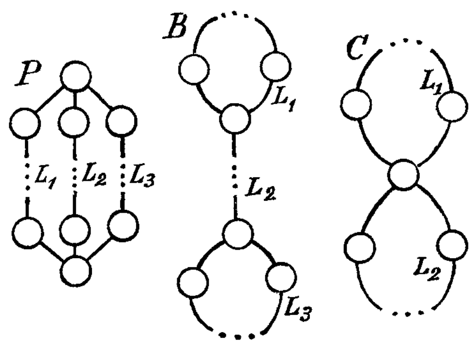


**Fig. 3**

We first deduce that the graphs we are looking for cannot be of type $C$ (see Fig. 3). Namely, by Corollary 2(c), any graph of type $C$ has its largest eigenvalue greater than $4/\sqrt{3}$. By Corollary 1(b), graphs of type $P$ attain the minimum largest eigenvalue if all paths are nearly equal. With these graphs, it is less than $4/\sqrt{3}$ for $n = 8$ (number of vertices). Since the largest eigenvalue decreases with $n$ the assertion is true for $n > 8$. For $n < 8$, we can easily check it. So we have only one candidate of type $P$ to match with the candidates of type $B$.

THEOREM 3. *Let $G = P(l_1, l_2, l_3)$ and $H = B(l_1, l_2, l_3)$ be the graphs suggested by Fig. 3. If $l_1 = l_3$, then $\rho(G) = \rho(H)$.*

*Proof.* Since the cycles in $H$ are of the same lenght, and also equal in lenght to two paths from $G$, we arrive to the same systems of difference equations, with the same boundary conditions. So the proof follows immediately.

*Remark* 3. The graph $H = B(l_1, l_2, l_3)$ with $l_1 \neq l_3$, cannot be as easily examined as the graphs from the previous section.

From the theorem above, it follows that some graph of type $B(l_1, l_2, l_3)$ with $l_1 = l_3$ may be as one of the graph we are looking for; otherwise, we get two graphs as solutions. Some experiments carried on the expert system "GRAPH" (see [9]), have supported the latter alternative. We hope to settle this problem in the near future.

**Acknowledgement.** We are especially grateful to D. M. Cvetković for drawing our attention to reference [8]; we are also indebted to Z. Radosavljević for making some useful comments.

REFERENCES

[1] F. Harary, *Graph Theory*, Addison Wesley, Reading, 1969.

[2] D. Cvetković: M. Doob, H. Sachs, *Spectra of Graphs, Theory and Applications*, Deutscher Verlag der Wissenschaften-Academic Press, Berlin-New York, 1980.

[3] L. Lovasz, J. Pelikan, *On the eigenvalues of trees*, Period. Math. Hungar. **3** (1973), 175–182.

[4] D. M. Cvetković: *Problem* 3*, in: Graphs, Hypergraphs and Applications*, Proc. Conf. Graph Theory, held in Eyba, October 1984, ed. H. Sachs, Teubner Verlagsgessellschaft, Leipzig 1985.

[5] A. J. Hoffman: *On limit points of spectral radii of non-negative symmetric integral matrices, in: Graph Theory and Applications.* Springer-Verlag, Berlin-Heidelberg-New York, pp. 165–172.

[6] A. J. Hoffman, J. H. Smith, *On the spectral radii of topologically equivalent graphs*, in: *Recent Advances in Graph Theory* (Proceedings of the Symposium held in Prague, June 1974, ed. M. Fiedler), Academia, Prague, 1975, pp. 273–281.

[7] P. Rowlinson, *A deletion-contraction algorithm for the characteristic polynomial of a multigraph*, to appear.

[8] Q. Li, K. Q. Feng, *On the largest eigenvalue of graphs*, (Chinese), Acta Math. Appl. Sinica **2** (1979), 167–175.

[9] D. M. Cvetković, L. L. Kraus, S. K. Simić, *Discussing graph theory with a computer, Implementation of the algorithms I*, Univ. Beograd Publ. Elektrotehn. Fak. Ser. Fiz. **716–734** (1981), 100–104.

Elektrotehnički fakultet
11000 Beograd
Jugoslavija

(Received 27 02 1986)