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ON SPECTRUM AND PER-SPECTRUM OF GRAPHS

Mieczysław Borowiecki

Abstract. We show that spectrum and per-spectrum of a graph G is $[x_1, \dots, x_n]$ and $[ix_1, \dots, ix_n]$, respectively,, iff G is a bipartite graph without cycles of length k, $k = 0 \pmod 4$).

Let $G = (V, E)$ be a finite graph without loops or multiple edges. Suppose the vertex set $V = \{v_1, \dots, v_n\}$. The adjacency matrix $A(G) = [a_{ij}]$ of G is the n by n matrix defined by

$$a_{ij} = \begin{cases} 1, & \text{if } \{v_i, v_j\} \in E, \\ 0, & \text{otherwise.} \end{cases}$$

By the characteristic (permanental) polynotnial of a graph G, written $F(G, x) = \sum_{i=0}^n a_i x^{n-i}$ $\left(f(G, z) = \sum_{i=1}^n b_i z^{n-i}\right)$, we mean the characteristic (permanental) polynomial of the adjacency matrix of G. If two graphs have the same characteristic (permanental) polynomial, they will be called cospectral (per-cospectral). All definitions and symbols not presented above can be found in [**3**] or [**4**].

Problem. Let $G = (V, E)$ be an n-vertex graph with spectrum $S(G) = [x_1, \dots, x_n]$. Characterize all graphs which have a pure imaginary per-spectrum $pS(G)$ of the form $[ix_1, \dots, ix_n]$.

Let us denote the class of all graphs with this property by $\mathcal{G}^*$. For trees the following holds ([**1**], [**5**]):

Proposition 1. *Let T_1 and T_2 be two nonisomorphic trees. Then T_1 and T_2 are per-cospectral if and only if T_1 and T_2 are cospectral.*

In 1978 (during a meeting in Hirschbach) H. Sachs noticed that in view of Proposition 1 any tree is in $\mathcal{G}^*$. In [**2**] the following conjecture was formulated:

Conjecture. $G \in \mathcal{G}^*$ if and only if G is a forest (each component of G is a tree).

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Unfortunately, the conjecture is false. The smallest counterexample is C_6:

$$F(C_6,x) = x^6 - 6x^4 + 9x^2 - 4, \qquad S(C_6) = [2,1,1,-1,-1,-2] \text{ and}$$
$$f(C_6,z) = z^6 + 6z^4 + 9z^2 + 4, \qquad pS(C_6) = [2i,i,i,-i,-i,-2i].$$

A natural question is whether one (characteristic or permanental) or both of these polynomials can distinguish nonisomorphic graphs. Proposition 1 shows that, at least for trees, the permanental polynomial distinguishes nothing which has not already been distinguished by the characteristic polynomial. An infinite class of graphs which are cospectral and per-cospectral simultaneously is described in [**1**], [**2**].

In the present paper we will give a solution of the mentioned problem and, as a corollary, an extension of Proposition 1.

We need some auxiliary results:

(1)

Let G be a graph with $F(G,x) = \sum_{i=0}^{n} a_i x^{n-i}$ and $f(G,z) = \sum i = 0^n b_i z^{n-i}$.

Then

(a) [6] $$a_i = \sum_{U_i \subset G} (-1)^{p(U_i)} 2^{c(U_i)}, \quad i = 1,2,\ldots,n,$$

(a) [1] $$b_i = (-1)^i \sum_{U_i \subset G} 2^{c(U_i)}, \quad i = 1,2,\ldots,n,$$

The summation is taken over all subgraphs U_i on i vertices whose components are circuits or K_2 (the subgraphs U_i will be called basic figures), $p(U_i)$ is the number of components of U_i, $c(U_i)$ is the number of components of U_i which are cycles of length ≥ 3.

(2) [1]: For a fixed i, $|a_i| = |b_i|$ if and only if all basic figures $U_i \subset G$ with exactly i vertices have the same parity of components.

(3) [1], [3]: G is bipartite if and only if $b_i = 0 \ (-a_i)$ for all odd i.

THEOREM. *$G \in \mathcal{G}^*$ if and only if G is a bipartite graph without cycles of length k, $k \equiv 0 \pmod 4$.*

Proof. Let $G \in \mathcal{G}$. From the form of the spectrum and the per-spectrum of G and by (3) it follows that G must be bipartite. Now suppose, G contains a cycle C_{4t} $(t \geq 1)$. Then G has at least two basic figures on $4t$ vertices with the different parity of components. Namely, $U"_{4t} = K_2 \cup \cdots \cup K_2$ ($2t$ times) and $U''_{4t} = C_{4t}$. By (2), $|a_{4t}| < |b_{4t}|$. Therefore, the spectrum and the per-spectrum of G can not be of the assumed form.

Conversely, suppose G is a bipartite graph without cycles of length k, where $k \equiv 0 \pmod 4$. By (2) and (3) it suffices to show that for a fixed i all basic figures

$U_i \subset G$ have the same parity of components. Assume U_i' and U_i'' to be two different basic figures on i vertices. Let U_i' has components: $C_{4k_1+2}, \dots, C_{4k_{s'}+2}, K_2, \dots, K_2$ ($K_2 - r'$ times), and U_i'': $C_{4l_1+2}, \dots, C_{4l_{s''}+2}, K_2, \dots, K_2$ ($K_2 - r''$ times). From above it follows that

$$(4k_1+2) + \cdots + (4k_{s'}+2) + 2r' = i = (4l_1+2) + \cdots + (4l_{s''}+2) + 2r''.$$

Then

(a) $$2(k_1 + \cdots + k_{s'}) + s' + r' = 2(l_1 + \cdots + l_{s''}) + s'' + r''.$$

It is easy to see that $s'+r'$ and $s''+r''$ are the numbers of components of U_i' and U_i'' respectively. Then (a) implies that U_i' and U_i'' have the same parity of components. And Theorem is proved.

From Theorem, as a corollary, we have an extension of Proposition 1:

COROLLARY. *If G_1 and G_2 are bipartite graphs without cycles of length k, $k \equiv 0 \pmod 4$, then G_1 and G_2 are per-cospectral if and only if G_1 and G_2 are cospectral.*

In [**2**] we formulated the following

Problem. Characterize those graphs which have pure imaginary per-spectrum.

From the above considerations and since C_4 has pure imaginary per-spectrum it follows that graphs which satisfy the above problem form a proper subclass of bipartite graphs, and they include the class $\mathcal{G}^*$ as a proper subclass.

In [**2**] we gave a construction of graphs with $n \geq 11$ vertices which are cospectral and per-cospectral simultaneously. Unfortunately, these graphs have a cutvertex. A pair of 2-connected graphs which are cospectral and per-cospectral is known to the author. But we have the following

Question. Are there, for any natural number k, graphs G, H which are k-connected and which form a pair of cospectral and per-cospectral graphs?

For $k = 1, 2$ the answer is *yes.*

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Higher College of Engineering
Department of Mathematics
65-246 Zielona Góra
Podgórna 50, Poland

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