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INTEGRABILITY OF TENSOR STRUCTURES OF ELECTROMAGNETIC TYPE

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Abstract. We study characterizations of the integrability of G-structures defined by tensor fields of elektromagnetic type.

1. Introduction. In [**3**] were considered the G-structures defined by a (1,1) tensor $\tilde{J}$ on a diffenertiable manifold M^n such that

$$(\tilde{J}^2 - f^2)(\tilde{J}^2 + g^2) = 0,$$

where f, g are C^∞ functions on M^n nowhere zero. This situation generalizes that of Hlavaty [**4**] and Mishra [**7**]. They consider the so called elektromagnetic tensor fields (of first class) on a 4-manifold which is the space-time of General Relativity. In [**3**] it was proved that the G-structure P defined by such a tensor field $\tilde{J}$ is identical to the G-structure defined by a (1, 1) tensor field J that satisfies the same conditions as $\tilde{J}$ but with $f = g = 1$, an so we have $J^4 = 1$.

On the other hand, that situation generalizes also the almost product and almost complex structures simultaneously. In [**9**] the family of linear connections that parallelize J (and an adapted metric also) is given. Connections partially adapted to such a structure are studied in [**11**]. In this note we study several characterizations and conditions of integrability of the (G-stucture defined by the) tenzor field $\tilde{J}$.

Thus, we consider the following situation:

Let M^n be a differentiable manifold and $\tilde{J}$ a (1,1) tensor field such that:

a) $(\tilde{J}^2 - f^2)(\tilde{J}^2 + g^2) = 0$, where f, g are C^∞ functions on M^n with f, g nowhere null;

b) The characteristic polynomial of $\tilde{J}$ is $(x - f)^{r_1}(x + f)^{r_2}(x^2 + g^2)^s$, where r_1, r_2, s are constants greater than or equal to 1 such that $r_1 + r_2 + 2s = n$. Then,

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since J which satisfies a) and b), but with $f = g = 1$, defines the same G-structure P as $\tilde{J}$ (not an associated G-structure, but exactly the same P see [**3**]), we can characterize the integrability of P in terms of J.

2. Integrability in terms of the Nijenhuis tensor. We denote from now on by $X, Y, \ldots$, vectors fields on M^n. We consider the complementary projection operators $l = (J^2 + 1)/2$, $l_3 = (1 - J^2)/2$, which verify

$$Jl = lJ;\ J^2 l = l,\ Jl_3 = l_3 J,\ J^2 l_3 = -l_3;$$

denote by L and L_3 the corresponding distributions, and put $L = L_1 \oplus L_2$, where L_1 and L_2 are distributions corresponding to the projectors l_1 and l_2 on L given by the eigenvalues $+1$ and -1 of $J\,|_L$. Let us decompose the Nijenhuis tensor of in the followig manner:

$$\begin{aligned} N(X,Y) = lN(lX,lY) + l_3N(lX,lY) + N(lX,l_3Y) \\ + N(l_3X,lY) + lN(l_3X,l_3Y) + l_3N(l_3X,l_3Y). \end{aligned}$$

Then we have the following

PROPOSITION 2.1. a) *L is integrable iff* $(\forall X,Y) l_3N(lX,lY) = 0$*;*

b) *L_3 is integrable iff* $(\forall X,Y) lN(l_3X,l_3Y) = 0$*;*

c) *If L is integrable, the almost product structure defined by $J\,|_L$ on each integral manifold of L is integrable iff* $(\forall X,Y) N(lX,lY) = 0$*;*

d) *If L_3 is integrable, the almost complex structure defined by $J\,|_{L_3}$ on each integral manifold of L_3 is integrable iff* $(\forall X,Y) N(l_3X,l_3Y) = 0$.

Proof. a) $N(lX,lY) = [JlX,JlY] - J[JlX,lY] - J[lX,JlY] + J^2[lX,lY]$. Thus, if L is integrable, each bracket is an element of L and so $l_3N(lX,lY) = 0$. Conversely, suppose now that $l_3N(lX,lY) = 0$; then we obtain easily:

$$\begin{aligned} & l_3N(JlX,JlY) + Jl_3N(JlX,lY) + Jl_3N(lX,JlY) \\ =& 3l_3[lX,lY] + l_3(N(lX,lY) - J^2[lX,lY]) \\ =& 4l_3[lX,lY] + l_3N(lX,lY), \end{aligned}$$

and since by the hypothesis $l_3N(lX,lY) = 0$, L is integrable;

b) Analogous to a), if we consider now

$$lN(Jl_3X,Jl_3Y) + JlN(Jl_3X,l_3Y) + JlN(l_3X,Jl_3Y);$$

c) If L is integrable, then $J\,|_L$ induces on each integral manifold of L an almost product structure. As such a structure is integrable iff its Nijenhuis tensor is zero, that is, $N_{J_{|L}}(lX,lY) = 0$, and since $N_{J_{|L}}(lX,lY) = N(lX,lY)$, we obtain c);

d) Similar to c). $\square$

Definition 2.2. We say that J is *partially integrable* iff L and L_3 are integrable, and also the almost product and almost complex structure induced by J on the integral manifolds of L and L_3, respectively.

Thus J is partially integrable iff $N(X,Y) = N(lX, l_3Y) + N(l_3X, lY)$.

So, we consider now the condition $N(lX, l_3Y) = 0$. Since the Lie derivative $L_Y J$ verifies by definition $(L_Y J)X = J[X,Y] - [JX,Y]$, we deduce:

a) $N(lX, l_3Y) = J(L_{l_3Y}J)lX - (L_{Jl_3Y}J)lX$;

b) $N(l_3X, lY) = J(L_{lY}J)l_3X - (L_{JlY}J)l_3X$;

and from these expressions it is immediate that:

Proposition 2.3. *$lN(lX, l_3Y) = 0$ (resp. $l_3N(lX, l_3Y) = 0$) for every X, Y iff $l(L_{l_3Z}J)l = 0$ (resp. $l_3(L_{lZ}J)l_3 = 0$ for every Z.*

Corollary 2.4. *$N(lX, l_3Y) = 0$ iff $l(L_{l_3Z}J)l = l_3(l_{lZ}J)l_3 = 0$, for every X, Y, Z.*

Now, we have

Theorem 2.5. *J is integrable iff $N_J = 0$.*

Proof. J is integrable iff for every $x \in M^n$, there exists a heighbourhood U of x and a coordinate system in $U, \{x^i\}$, such that the basis $\{\partial/\partial x^i\}$, $i = 1, \ldots, n$ is adapted in U. That is, J can be expressed as a linear combination of products $\partial/\partial x^i \otimes dx^j$ with constant coefficients, and so, trivially, $N = 0$.

Conversely, suppose $N = 0$. By a)and b) of Prop. 2.1, L and L_3 are integrable. Thus, for each $x \in M^n$ there exists a chart centered at $x, (U, \varphi)$, with coorinates $\{x^i, y^a\}, i = 1, \ldots, r_1 + r_2$, $a = 1, \ldots 2s$, such that

$$\partial/\partial x^i \in L, \ \partial/\partial y^a \in L_3.$$

So, in the local basis $\{\partial/\partial x^i, \ \partial/\partial y^a\}$, J has a matrix of the from

$$J = \begin{pmatrix} J^i_j & 0 \\ 0 & J^a_b \end{pmatrix};$$

that is, $J = J^i_j \partial/\partial x^i \otimes dx^j + J^a_b \partial/\partial y^a \otimes dy^b$. Moreover, $l = \partial/\partial x^i \otimes dx^i$, and $l_3 = \partial/\partial y^a \otimes dy^a$. Thus,

$$L_{\partial/\partial y^c} J = \frac{\partial J^i_j}{\partial y^c} \frac{\partial}{\partial x^i} \otimes dx^i + \frac{\partial J^a_b}{\partial y^c} \frac{\partial}{\partial y^a} \otimes dy^b.$$

Hence

$$l(L_{\partial/\partial y^a} J)l = \frac{\partial J^i_j}{\partial y^a} \frac{\partial}{\partial x^i} \otimes dx^j.$$

So, Corollary 2.4 implies $\partial J^i_j/\partial y^a = 0$ (and analogously $\partial J^a_b/\partial x^i = 0$).

Consider now the integral manifold L_x of L, of coordinates $y^a = 0$. Since the almost product structure on L_x is integrable, there exist coordinate functions u^i in a neighbourhood of $x \in L_x$ in L_x such that

$$\partial/\partial u^i \in L_1,\ i = 1, \ldots, r_1,\ \partial/\partial u^i \in L_2,\ i = r_1 + 1, \ldots, r_1 + r_2,$$

where these fields are considered in the regular submanifold $L_x \cap U$.

We define new coordinates in a neighbourhood W of x in U: put, for $x' \in W$,

$$u^i(x') = u^i(\varphi^{-1}(x^1(x'), \ldots, x^{r_1+r_2}(x')), 0, \ldots, 0),\ \bar{y}^a(x') = y^a(x).$$

Then, for the new coordinate system $\{u^i, y^{-a}\}$ we have

$$\frac{\partial}{\partial x^i} = \frac{\partial u^j}{\partial x^i}\frac{\partial}{\partial u^j},\ dx^i = \frac{\partial x^i}{\partial u^j} du^j,\ \text{ where } \frac{\partial}{\partial \bar{y}^a}\Big(\frac{\partial u^i}{\partial x^i}\Big) = \frac{\partial}{\partial \bar{y}^a}\Big(\frac{\partial x^i}{\partial u^j}\Big) = 0$$

This is a new coordinate system adapted to L and L_3, and we have

$$J = \overline{J}^i_j \frac{\partial}{\partial u^i} \otimes du^j + \overline{J}^a_b \frac{\partial}{\partial \bar{y}^a} \otimes d\bar{y}^b,\ \text{ with } \frac{\partial \overline{J}^i_j}{\partial \bar{y}_a} = 0.$$

But for the points of coordinates $\bar{y}^a = 0$ we have by construction (see 2.1)

$$\overline{J}^i_j = \begin{pmatrix} I_{r_1} & 0 \\ 0 & -I_{r_2} \end{pmatrix}.$$

Hence, in certain neighbourhood of x we also have the same matrix expression for $\overline{J}^i_J$.

Similarly, since the structure defined by J in L_{3x} is almost complex, a change of coordinates analogous to the previous one gives for $\overline{J}^a_b$ the expression

$$\begin{pmatrix} 0 & -I_s \\ I_s & 0 \end{pmatrix}.$$

In other words, we have deduced that J is integrable. □

3. Integrability in terms of a linear connection. Now, let ∇ be a linear connection without torsion on M^b and let Q be the (1, 2) tensor field on M^n defined by $Q(X,Y) = \{(\nabla_{JY} J)X + J(\nabla_Y J)X + 2J(\nabla_X J)Y\}/4$.

We define a new connection D by means of the expression

$$D_X Y = l\nabla_X lX - l\nabla_{lY} l_3 X + l_3 \nabla_X l_3 Y - l_3 \nabla_{l_3 Y} lX + lQ(lX, lY) - l_3 Q(l_3 X, l_3 Y).$$

It is easily proved that:

$$\begin{aligned}
i)\ & D_X lY = l\nabla_X lY - l\nabla_{lX} l_3 X + lQ(lX, lY);\\
ii)\ & D_X l_3 Y = l_3 \nabla_X l_3 Y - l_3 \nabla_{l_3 Y} lX - l_3 Q(l_3 X, l_3 Y);\\
iii)\ & D_{lX} lY = l\nabla_{lX} lY + lQ(lX, lY);\\
iv)\ & D_{l_3 X} lY = l\nabla_{l_3 X} lY - l\nabla_{lY} l_3 X;\\
v)\ & D_{lX} l_3 Y = l_3 \nabla_{lX} l_3 Y - l_3 \nabla_{l_3 Y} lX;\\
vi)\ & D_{l_3 X} l_3 X = l_3 \nabla_{l_3 X} l_3 Y - l_3 Q(l_3 X, l_3 Y);\\
vii)\ & D_X l = D_X l_3 = 0.
\end{aligned} \tag{3.1}$$

So we have

PROPOSITION 3.1. *The torsion T of D has the expression*

$$T(X,Y) = 1/4\{lN(lY,lX) + l_3N(l_3N(l_3X,l_3Y)\} - l[l_3X,l_3Y] - l_3[lX.lY].$$

Proof. Immediate from the expression for D and Q, applying that ∇ is torsionless, and proving that $Q(X,Y) - Q(Y,X) = N(Y,X)/4$ □

From that we obtain

COROLLARY 3.2. $(\forall X,Y) T(lX,l_3Y) = 0$.

We now prove

LEMMA 3.3. a) *The distribution L is integrable iff $l_3T(lX,lY) = 0$;*

b) *The distribution L_3 is integrable iff $lT(l_3X,l_3Y) = 0$;*

c) *If L is integrable, then the almost product structure induced by $J|_L$ on each integral manifold of L is integrable iff $lT(lX,lY) = 0$;*

d) *If L_3 is integrable, then the almost complex structure induced by $J|_{L_3}$ on each integral manifold of L_3 is integrable iff $l_3T(l_3X,l_3Y) = 0$;*

e) *J is partially integrable iff $T(X,Y) = 0$;*

f) $lN(JlX,lY) = lN(lX,JlY)$; *g*) $lT(JX,lY) = lT(lX,JY)$;

h) $l_3N(Jl_3X,l_3Y) = l_3N(l_3X,Jl_3Y)$; *i*) $l_3T(JX,l_3Y) = l_3T(l_3X,JY)$;

j) $(D_{l_3X}J)l_3Y = 0$; *k*) $(D_{lX}J)lY = 0$.

Proof. a) If suffices to prove $l_3T(lX,lY) = -l_3[lX,lY]$;

b) analogous to a); c) it suffices to consider $lT(lX,lY) = lN(lY,lX)/4$ and a), c) of Prop. 2.1; d) it is deduced from $l_3T(l_3X,l_3Y) = l_3N(l_3X,l_3Y)/4$, and b), d) of Prop. 2.1.;

e) from Cor. 3.2 we obtain $T(X,Y) = T(lX,lY) + T(l_3X,l_3Y)$. If J is partially integrable, from a) and c) we deduce $T(lX,lY) = 0$ and from b) and d) that $T(l_3X,l_3Y) = 0$. Hence $T(X,Y) = 0$. Conversely, $T(X,Y) = 0$ implies $T(lX,lY) = T(l_3X,l_3Y) = 0$ and thus $lT(lX,lY) = l_3T(lX,lY) = lT(l_3X,l_3Y) = l_3T(l_3X,l_3Y) = 0$.

From these equalities and from a), b), c), d) we deduce that J is partially integrable; f) the proof is immediate and moreover, as a consequence, we obtain $lN(JlX,JlY) = lN(lX,lY)$; g) $lT(JX,lY) = lN(lY,JlX)/4$ and $lT(JX,lX) = lN(JlY,lX)/4$, and from f) we obatin the result; h) the proof is analogous to that of f) and we deduce here $l_3N(Jl_3X,Jl_3Y) = -l_3N(l_3X,l_3Y)$;

i) we have $l_3T(JX,l_3Y) = l_3N(Jl_3X,l_3Y)/4$ and $l_3T(l_3X,JY) = l_3N(l_3X,Jl_3Y)/4$, and the conclusion follows from h);

j) from (3.1), vi) we have $D_{l_3X}Jl_3Y = l_3\nabla_{l_3X}Jl_3Y - l_3Q(l_3X,Jl_3Y)$ and $JD_{l_3X}l_3Y = Jl_3\nabla_{l_3X}l_3Y - l_3JQ(l_3X,l_3Y)$. Substracting we obtain

$$\begin{aligned}(D_{l_3X}J)l_3Y &= l_3(\nabla_{l_3X}J)l_3Y + l_3\{J(\nabla_{Jl_3Y}J)l_3X + J^2(\nabla_{l_3X}J)l_3X \\ &+ 2J^2(\nabla_{l_3X}J)l_3Y - (\nabla_{J^2l_3Y}J)l_3X - J(\nabla_{Jl_3X}J)l_3X \\ &- 2J(\nabla_{l_3X})Jl_3Y\}/4 = l_3(\nabla_{l_3X}J)l_3Y - l_3(\nabla_{l_3X}J)l_3Y = 0;\end{aligned}$$

k) analogous to that of j), by using (3.1), iii). □

THEOREM 3.4. *J is integrable iff there exists a linear conection without torsion that parallelzes J. If J is integrable, then D gives an explicit example of such a connection.*

Proof. Suppose J integrable. Then for the earlier connection D we have: 1) D is torsionless, and 2) $DJ = 0$. Indeed, if J is integrable then it is partially integrable and, from e) of Lemma 3.3. we obtain 1). On the other hand, if J is integrable, $N(X,Y) = 0$, and since from (3.1), v) we have

$$(D_{JlX}J)l_3Y + (D^{\cdot}_{lX}J)Jl_3Y = l_3N(lX, l_3Y) = 0,$$

we deduce

$$(D^{\cdot}_{JlX}J)l_3Y = -(D_{lX}J)Jl_3Y. \tag{3.2}$$

Substituting Y by JY we have

$$(D_{JlX}J)l_3JY = -(D_{lX}J)J^2l_3Y = (D_{lX}J)l_3Y, \tag{3.3}$$

and, if in (3.2) we substitute X by JX, we have

$$(D^{\cdot}_{JlX}J)Jl_3Y = -(D_{J^2lX}J)l_3Y = -(D^{\cdot}_{l}J)l_3Y. \tag{3.4}$$

From (3.3) and (3.4) we deduce

$$(D^{\cdot}_{lX}J)l_3Y = 0. \tag{3.5}$$

From (3.1), iv) we have $lN(l_3X, lY) = (D_{Jl_3X}J)lY + (D_{l_3X}J)JlY$; if J is integrable we obtain analogously

$$(D_{l_3X}J)lY = 0. \tag{3.6}$$

But in j) and k) of Lemma 3.3. we have

$$(D_{l_3X}J)l_3Y = 0 \text{ and } (D_{lX}J)lY = 0 \tag{3.7}$$

Hence, from (3.5) (3.6) and (3.7) follows 2).

Conversly, suppose now that there is a linear connection ∇ without torsion such that $\nabla J = 0$. If we consider Q from ∇ as before, we see that $Q = 0$. But in the proof of the Prop. 3.1 we have seen that $Q(X,Y) - Q(Y,X) = N(Y,X)/4$; that is, J is integrable.

Remark. As is well known, Lehmann-Lejeune [**5**] proves that, for 0-deformable (1,1) tensor fields, the integrability is equivalent to the existence of a torsionless structural *local* connection. In our case, we have a global connection and we also give its explicit expression when J is integrable.

4. Integrability in terms of the structure tensor. We have now at disposal two criteria of integrability of the G-structure P defined by $\tilde{J}$. The first one in terms of the Nijenhuis tensor of the field J, the second one in terms of a linear connection. A third criterion is that which expresses the integrability in terms of the Guillemin stucture tensors [**2**].

The field $\tilde{J}$ is not 0-deformable, but the associted field J, which defines the same structure P, is 0-deformable and so, we can anew characterize the integrability of P in terms of J; but from the results of Lehmann-Lejeune [**5**] it suffices to consider, in this case, the Chern-Ehresmann-Bernard tensor [**1**] and have the equivalence of the integrability with nullity of the 1-st structure tensor of P, as we express in the final theorem.

5. Integrability in terms of prolongations and complete lifts. Now, we consider of the one hand the complete lift J^c of J in the sense of Yano-Kobayashi [**12**], whic is a (1,1) tensor field on TM^n defined from J and, on the other hand, the canonical prolongation $\hat{J}$ of J in the sense of Morimoto [**8**], which is also a (1,1) tensor field on TM^n. Firstly, we have the following:

PROPOSITION 5.1. *The canonical prolongation to TN^n of the $(J^4 = 1)$-structure J is a $(J^4 = 1)$-structure $\hat{J}$ which coincides with the complete lift J^c of J.*

Proof. The structural group G correspoding to J (and $\tilde{J}$) is that of matrices of the form [**3**]

$$\left[\begin{array}{cc|cc} A & 0 & & \\ 0 & B & \multicolumn{2}{c}{\smash{0}} \\ \hline & & C & -D \\ \multicolumn{2}{c|}{\smash{0}} & D & C \end{array}\right]$$

where $A \in Gl(r_1, \mathbf{R})$, $B \in Gl(r_2, \mathbf{R})$, $C + iD \in Gl(s, \mathbf{C})$, $r_1 + r_2 + 2s = n$. If we denote $\hat{G}$ the structural group of the canonical prolongation $\hat{P}$ of P defined by J (and $\tilde{J}$), it has as elements the $\hat{g}$ obtained by means of

$$\hat{g} = j_n(\{g, X\}), \quad g \in G, \quad X \in \mathbf{g},$$

where $\mathbf{g}$ denotes the Lie algebra of G, X the translated $R_{g^{-1}*}Y$, for a certain $Y \in T_gG$, and j_n the imbedding $j_n : TGl(n, \mathbf{R}) \to Gl(2n, \mathbf{R})$.

More precisely,

$$j_n(\{g, X\}) = \begin{bmatrix} g & 0 \\ Xg & g \end{bmatrix} = \left[\begin{array}{c|c|c|c} \begin{matrix} A & 0 \\ 0 & B \end{matrix} & 0 & 0 & 0 \\ \hline 0 & \begin{matrix} C & -D \\ D & C \end{matrix} & 0 & 0 \\ \hline \begin{matrix} \alpha & 0 \\ 0 & \beta \end{matrix} & 0 & \begin{matrix} A & 0 \\ 0 & B \end{matrix} & 0 \\ \hline 0 & \begin{matrix} \gamma & -\delta \\ \delta & \gamma \end{matrix} & 0 & \begin{matrix} C & -D \\ D & C \end{matrix} \end{array}\right] \tag{5.1}$$

since

$$Xg = \left[\begin{array}{c|c} \begin{matrix} M & 0 \\ 0 & N \end{matrix} & 0 \\ \hline 0 & \begin{matrix} P & -Q \\ Q & P \end{matrix} \end{array}\right] \left[\begin{array}{c|c} \begin{matrix} A & 0 \\ 0 & B \end{matrix} & 0 \\ \hline 0 & \begin{matrix} C & -D \\ D & C \end{matrix} \end{array}\right] = \left[\begin{array}{c|c} \begin{matrix} MA & 0 \\ 0 & NB \end{matrix} & 0 \\ \hline 0 & \begin{matrix} PC-QD & -PD-QC \\ QC+PD & -QD+PC \end{matrix} \end{array}\right]$$

where $M \in \mathcal{M}(r_1; \mathbf{R}),\ N \in \mathcal{M}(r_2; \mathbf{R}),\ P + iQ \in \mathcal{M}(s, \mathbf{C})$.

It is immediate that $j_n(\{g, X\})$, belongs to the matrix group $Gl(2r_1, \mathbf{R}) \times Gl(2r_2, \mathbf{R}) \times Gl(2s, \mathbf{C})$, by means of a convenient rearrangement of the boxes of the matrix (5.1). Hence, we have the structural group od a $(J^4 = 1)$-structure on TM^n.

On the other hand, for a given local coordinate system $\{U, x^1, \dots, x^n\}$ on M^n, and a section σ of the principal bundle of frames FM^n on U, expressed as

$$\sigma(x) = \Big(\dots, \sum_{i=1}^{n} \sigma_j^i(x) \frac{\partial}{\partial x^i}\Big|_x, \dots\Big),\ \ x \in U,$$

Morimoto [**8**] proves that $\tilde{\sigma} = j_{M^n} \cdot T\sigma$ (where j_{M^n} is the canonical embedding $j_{M^n} : TFM^n \to FTM^n$), is a section of FTM^n on TU, which can be expressed as

$$\tilde{\sigma}\Big(\sum_{i=1}^{n} y^i \frac{\partial}{\partial x^i}\Big|_x\Big) = \Big(\dots, \sum_{i-1}^{n} \sigma_j^i(x) \frac{\partial}{\partial x^i}\Big|_x + \sum_{i,k=1}^{n} \frac{\partial \sigma^i(x)}{\partial x^k} y^k \frac{\partial}{\partial y^i}\Big|_X \dots,$$
$$\sum_{i=1}^{n} \sigma_j^i(x) \frac{\partial}{\partial y^i}\Big|_X, \dots\Big)$$

where $\{x^1, \dots, x^n,\ y^1, \dots, y^n\}$ is the local coordinate system induced in TU, and $X = \sum\limits_{i=1}^{n} y^i \frac{\partial}{\partial x^i}\Big|_x \in TU$.

We now consider the diagram

$$\begin{tikzcd}
FV & TFM^n \arrow[l] \arrow[r, "j_{M^n}"] & FTM^n \arrow[ddl] \\
P \arrow[u, hook] \arrow[d, "\pi"] & TP \arrow[l] \arrow[u, hook] \arrow[d, "T\pi"] & \\
M^n \arrow[u, bend left, "\sigma"] & TM^n \arrow[l] \arrow[u, bend left, "T\sigma"] \arrow[uur, bend right, "\tilde{\sigma} = j_{M^n} \cdot T\sigma"'] &
\end{tikzcd}$$

with the explained notations. Let again $\{U, x^1, \ldots, x^n\}$ a local coordinate system on M^n, and σ a section of the G-structure P on U; then $\tilde{\sigma} = j_{M^n} \cdot T\sigma$ is a section of the canonical prolongation $\hat{P}$ of P, since

$$\hat{\sigma}(TU) = j_{M^n} \cdot T\sigma(TU) \subset j_{M^n}(T(\sigma(U)) \subset j_{M^n}(TP) = \hat{P}.$$

Now, let $J_0 : \mathbf{R}^n \to \mathbf{R}^n$ be an automorphsm such that $J_0^4 = 1$.

From the diagram

$$\begin{array}{ccc} T_xM^n & \xrightarrow{J_x} & T_xM^n \\ {\scriptstyle\sigma(x)}\uparrow & & \uparrow{\scriptstyle\sigma(x)} \\ \mathbf{R}^n & \xrightarrow{J_0} & \mathbf{R}^n \end{array}$$

we define J_x as $J_x = \sigma(x) \cdot J_0 \cdot \sigma(x)^{-1}$.

Then $J : x \sim\to J_x$ is the $(J^4 = 1)$-structure associated to P globally defined. Indeed:

a) $J^4 = 1$. Immediate from $J_0^4 = 1$;

b) globaly defined: If $x \in U \cap U'$, where U, U' are coordinate neghbourhoods and σ' is a section of P on U', then $J'_x = \sigma'(x) \cdot J_0 \cdot \sigma'(x)^{-1}$, but since

$$\sigma'(x) = g(x) \cdot \sigma(x), \quad g(x) \in G.$$

we deduce

$$J'_x = g(x) \cdot \sigma(x) \cdot J_0 \cdot \sigma(x)^{-1} \cdot g(x)^{-1} = g(x) \cdot J_x \cdot g(x)^{-1}.$$

We now consider $TJ_0 : T\mathbf{R}^n \to T\mathbf{R}^n$. Since

$$TJ_0 = \begin{bmatrix} J_0 & 0 \\ 0 & J_0 \end{bmatrix}$$

we have $(TJ_0)^4 = 1$, and we define

$$\hat{J}(X) = \hat{\sigma}(X) \cdot TJ_0 \cdot \hat{\sigma}(X)^{-1}, \text{ for every } X \in TU.$$

It is clear that $\hat{J}^4 = 1$, and $\hat{J}$ is the $(J^4 = 1)$-structure on TM^n canonical prolongation of J, since $\hat{\sigma} = j_{M^n} \cdot T\sigma$.

On the other hand, we can choose as a basis of $T_X TM^n$ the set

$$\left\{ \frac{\partial}{\partial x^1}\Big|_X, \dots, \frac{\partial}{\partial x^n}\Big|_X, \frac{\partial}{\partial y^1}\Big|_X, \dots, \frac{\partial}{\partial y^n}\Big|_X \right\}.$$

Then, using the earlier expressions for $\hat\sigma$ and TJ_0, we obtain

$$\hat J(x) = \hat\sigma(x)\cdot TJ_0\cdot \hat\sigma(x)^{-1} = \begin{bmatrix} \sigma(x) & 0 \\ \partial\sigma(x) & \sigma(x) \end{bmatrix} \begin{bmatrix} J_0 & 0 \\ 0 & J_0 \end{bmatrix} \begin{bmatrix} \sigma(x) & 0 \\ \partial\sigma(x) & \sigma(x) \end{bmatrix}^{-1}$$
$$= \begin{bmatrix} J_x & 0 \\ \partial J_x & J_x \end{bmatrix},$$

which is precisely the formula of the complete lift J^c of J (see [**12**]), being

$$\partial J_x = \left(\sum_{k=1}^n y^k \frac{\partial J^i_j}{\partial x^k} \right)$$

□

But Morimoto [**8**] proves that a G-structure P is integrable if and only if the canonical prolongation $\hat P$ is integrable; hence we obtain.

Proposition 5.2. *Let (M^n, J) be a $(J^4 = 1)$-manifold. Then the following statements are equivalent:*

a) *The G-structure P defined by J is integrable;*

b) *The Nijenhuis tensor of the tensor $\hat J$ corresponing to the canonical prolongation $\hat P$ of P is zero;*

c) *The Nijenhuits tensor of the complete lift J^c of J is zero.*

6. J-Lie groups. Now, we consider a sufficient condition in order to a $(j^4 = 1)$-structure be integrable.

Let (M_1, J_1) and (M_2, J_2) be two $(J^4 = 1)$-manifolds. We say that a differentiable map $f : M_1 \to M_2$ is a J-map if and anly if the following diagram is commutative

$$\begin{array}{ccc} T_x M_1 & \xrightarrow{f_*} & T_{f(x)} M_2 \\ {\scriptstyle J_1}\downarrow & & \downarrow{\scriptstyle J_2} \\ T_x M_1 & \xrightarrow{f_*} & T_{f(x)} M_2 \end{array} \qquad \text{for every } x \in M_1.$$

Definition 6.1. We call *J-Lie group* a Lie Group G with a $(J^4 = 1)$-structure J such that the usual translations L_g and R_g are J-maps, for every $g \in G$.

Thus, we have

Proposition 6.2. *If G is a J-Lie group, then J is integrable.*

Proof. Since L_g and R_g are J-maps, we have $\mathrm{ad}\, g \cdot J = J\cdot \mathrm{ad}\, g$.

In particular for $g = \exp tX$, $X \in \mathbf{g}$, $t \in \mathbf{R}$, we have

$$\exp(\operatorname{Ad} tX) \cdot JY = J(\exp(\operatorname{Ad} tX)Y), \text{ for every } Y \in g. \tag{6.1}$$

Moreover, we obtain

$$\begin{aligned} \exp(\operatorname{Ad} tX)JY &= JY + t[X,JY] + t^2[X,[X,JY]]/2 + \cdots \\ J\exp(\operatorname{Ad} tX)Y &= JY + tJ[X,Y] + t^2 J[X,[X,Y]]/2 + \cdots . \end{aligned}$$

Hence, from (6.1) and taking the limit for $t \to 0$ we deduce $[X, JY] = J[X,Y]$, and also $J[X,Y] = -J[Y,X] = -[Y,JX] = [JX,Y]$.

Thus, it is immediate $N(X,Y) = 0$, $X, Y \in \mathbf{g}$. □

7. Characterizations of the integrability. Finally, according to the earlier results we can give the following:

THEOREM 7.1. *Let M^n be a differentiable manifold with a tensor field of electromagnetic type and class $\tilde{J}$. Then the following statements are equivalent:*

a) *The G-structure P defined by $\tilde{J}$ is integrable;*

b) *The Nijenhuis tensor of the associated tensor field J is zero;*

c) *There exists a linear torsionless connection which parallelizes J;*

d) *The structure tensor of P is zero;*

e) *The Nijenhuis tensor of the tensor field $\hat{J}$ corresponding to the canonical prolongation $\hat{P}$ of P is zero;*

f) *The Nijenhuis tensor of the complete lift J^c of J is zero; moreover,*

g) *If G is a J-Lie group, then J is integrable.*

When M^n is J-Kaehlerian [**10**], other conditions can be given.

We note that any linear connection which parallelizes $\tilde{J}$ does not exist.

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