PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 37 (51), 1985, pp. 107–112

ON THE FOCAL LOCUS OF A SUBMANIFOLD IN A RIEMANNIAN MANIFOLD OF CONSTANT SECTIONAL CURVATURE

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Abstract. We discuss the focal locus of a submanifold in a Riemannian manifold of constant sectional curvature. Focal points of submanifolds in Riemannian manifolds have been studied already in [**1, 4, 7**] and [**8**]. We consider the location of focal points along a geodesic, and we also obtain conditions under which the focal locus is a submanifold of given dimension.

1. Introduction

Let M be a C^∞ n-dimensional complete Riemannian manifold of constant sectional curvature K. Let L be a C^∞ m-dimensional submanifold imbedded in M. The orthogonal complement N_pL of the tangent space T_pL in T_pM will be called the normal space of L at p for $p \in L$. The union $N = \cup\{N_pL \mid p \in L\}$ of these normal spaces which is a subbundle of class C^∞ of the tangent bundle TM is called the normal bundle of the submanifold L.

Let $c : [0,a] \to M,\ a \in R$ be a normal geodesic with $c(0) = z \in L,\ \dot{c}(0) = v \in N_zL$. Let $Y(t),\ t \in [0,a]$ be a Jacobi field along c, then $Y(t)$ is an L-Jacobifield if [**1**]

$$\text{(i)} \qquad \langle Y(t), \quad \dot{c}(t)\rangle = 0,$$

$$\text{(ii)} \qquad Y(0) \in T_zL,$$

$$\text{(iii)} \qquad \nabla_{\dot{c}(0)}Y(0) + A_{\dot{c}(0)}Y(0) \in N_zL,$$

where $A_{\dot{c}(0)} : T_zL \to T_zL$ is the Weingarten map with a different sign convention from that of [**1**].

A point $c(t_0)$ on c for $t_0 \in (0,a]$, is called a focal point of L with respect to c if there is a non-trivial L-Jacobi field on c which vanishes at $t = t_0$. The set of such focal points is called focal locus of L in M.

AMS Subject Classification (1980): Primary 53 B 21, Secondary 53 C 40.

Let $E_1, \dots, E_{n-1} : [0, a] \to TM$ be parallel vector fields along c such that $(c(t),\ E_1(t), \dots, E_{n-1}(t))$ is an orthonormal base for $T_{c(t)}M$, then a Jacobi field in the space of constant sectional curvature K is given by

$$Y(t) = \sum_{i=1}^{n-1} (a_i \sin(\sqrt{K}t) + b_i \cos(\sqrt{K}t)) E_i(t), \ \text{ if } K > 0,$$
$$Y(t) = \sum_{i=1}^{n-1} (a_i t + b_i) E_i(t), \ \text{ if } K = 0,$$
$$Y(t) = \sum_{i=1}^{n-1} (a_i \sinh \sqrt{K}t) + b_i \cosh(\sqrt{K}t))\, E_i(t), \ \text{ if } K < 0,$$

(see e. g. [**3**]). Therefore the value of focal points depends on the value of K. The case when $K = 0$, has already been studied in [**5, 6**]. In this paper we will consider the remaining cases and will study the location of focal points by using the Jacobi field given in the above form.

2. Position of focal points along geodesic

Our first aim is to specify the from of Jacobi field given in [**3**], so as to give an L-Jacobi field. Assume now that T_zL is spanned by $E_1(0), \dots, E_m(0)$ and that these vectors are eigenvectors of the symetric map $A_{\dot{c}(0)}$.

Condition (i) for $Y(t)$ to be L-Jacobi field is safisfied by the structure of $Y(t)$. For condition (ii)

$$Y(0) = \sum_{i=1}^{m} b_i E_i(0) \in T_zL,$$

and therefore $b_{m+1}, \dots, b_{n-1} = 0$, if $(E_1, \dots, E_m)$ is a base for T_zL. For condition (iii) we have

$$\nabla_{\dot{c}(t)} Y(t) = \sum_{i=1}^{n-1} (a_i \sqrt{K} \cos(\sqrt{K}t) - b_i \sqrt{K} \sin(\sqrt{K}t)) E_i(t)$$

since E_i are parallel. Therefore

$$\nabla_{\dot{c}(0)} Y(0) = \sum_{i=1}^{n-1} a_i \sqrt{K} E_i(0)$$
$$A_{\dot{c}(0)} Y(0) = A_{\dot{c}(0)} \sum_{i=1}^{m} b_i E_i(0) = \sum_{i=1}^{m} b_i \lambda_i E_i(0),$$

where λ_i is the eigenvalue of the symmetric linear transformation $A_{\dot{c}(0)}$ corresponding to the eigenvector E_i. Therefore

$$\nabla_{\dot{c}(0)} Y(0) + A_{\dot{c}(0)} Y(0) = \sum_{i=1}^{n-1} a_i \sqrt{K} E_i + \sum_{i=1}^{m} b_i \lambda_i E_i \in N_zL$$

if $a_i\sqrt{K} + b\lambda_i = 0$ for $i = 1, \dots, m$. So, the general form of L-Jacobi field is

$$Y(t) = \sum_{i=1}^{m} a_i(\sin\sqrt{K}t) - \sqrt{K}/\lambda_i \cdot \cos(\sqrt{K}t))E_i + \sum_{i=m+1}^{n-1} a_i \sin(\sqrt{K}t)E_i$$

For finding the location of the focal points of L along c we may consider the following suitable L-Jacobi field of the above structure

$$Y(t) = a(\sin(\sqrt{k}t) - \sqrt{K}/\lambda_i \cdot \cos(\sqrt{K}t))E_i(t).$$

Let $c(t_i)$ be i-th focal point of L along c, then $Y(t_i) = 0$, i. e. $\sin(\sqrt{K}t_i) - \sqrt{K}/\lambda_i \cdot \cos(\sqrt{K}t_i) = 0$. Consequently $\tan\sqrt{K}t_i = \sqrt{K}/\lambda_i$.

Let t_1 be the smallest positive solution of the above equation. Then the first focal point of L along c occurs at $c(t_1)$. Let $t_1, \dots, t_m$ be the solutions of the above equation s.t. $0 < t_1 \le t_2 \dots \le t_p < \infty$. The eigenvalues of the symmetric linear transformation $A_{\dot{c}(0)}$ define the principal curvaures of L at z in the direction of $\dot{c}(0)$ by $\rho_i = t_i = 1/\sqrt{K}\arctan\sqrt{K}/\lambda_i$ where $\rho_i (i = 1, \dots, m)$ are principal radii of curvatures. Thus we have

THEOREM. *Let M be a C^∞ n-dimensional complete Riemannian manifold of constant sectional curvature $K \ne 0$, and let L be a C^∞ m-dimensional submanifold imbedded in M. Let $c : [0, a] \to M$ be a normal geodesic with $c(0) = z \in L$, $\dot{c}(0) = y \in N_zL$. Then the position of m-focal points of L along c are given by*

(1) $\quad 1/\sqrt{K} \cdot \arctan(\sqrt{K}/\lambda_i) = \rho_i, \quad \text{if } K > 0$

(2) $\quad 1/\sqrt{K} \cdot \operatorname{arctan\,h}(\sqrt{K}/\lambda_i) = \rho_i, \quad \text{if } K < 0, \quad i = 1, \dots, m.$

The m focal points along c will generate m subset of the focal locus in M.

3. To find the conditions for the focal locus to be a submanifold of given dimension 1

Let M be a Riemannian manifold. Consider first the case when $1 = n - 1$. Let $(u^1, \dots, u^m)$ be a local coordinate system in the neighbourhood U of a point $z \in L$ which is orthonormal at z, and a field $(w_1, \dots, w_{n-m})$ of orthonormal base of N_zL for $z \in U$. Consider the diffeomorphism $R^n \to N(L)$ given by

$$(u^1, \dots, u_m;\ \varphi^1, \dots, \varphi^{n-m-1}; t) \to q = t\sum_{i=1}^{n-m} \varphi^i w_i(u^1, \dots, u^m),$$

$q \in N(L)$. Since this diffeomorphism has no singular points, in order to study focal points we cosider $R^n \to N(L) \to M$ such that

$$(u_1, \dots, u^m; \varphi^1 \dots, \varphi^{n-m-1}; t) \to q \to \varepsilon(q) = r(u^1, \dots, u^m;\ \varphi^1, \dots, \varphi^{n-m-1}; t).$$

In this manner $t_i = 1/\sqrt{K} \cdot \arctan\sqrt{K}/\lambda_i$ can be written as $t_i = t_i(u^1, \dots, u^m; \varphi^1, \dots, \varphi^{n-m-1})$.

Consider now the curves in the manifold M which are given by

$$r(u^1, u_0^2, \dots, u_0^m;\ \varphi_0^1, \dots, \varphi_0^{n-m-1};\ t_0), \dots$$
$$r(u_0^1, \dots, u_0^m;\ \varphi_0^1, \dots, \varphi_0^{n-m-1};\ t_0), r(u_0^1, \dots, u_0^m;\ \dots, \varphi_0^{n-m-1};\ t)$$

where $u_0^1, \dots, u_0^m;\ \varphi_0^1, \dots, \varphi_0^{n-m-1};\ t_0$ are constants. Therefore the corresponding tangent vectors to these curves will be given by

$$\partial r/\partial u^1, \dots, \partial r/\partial u^m,\ \partial r/\partial \varphi^1, \dots, \partial r/\partial \varphi^{n-m-1},\ \partial r/\partial t = \dot{c}(t).$$

Let $t = t(u^1, \dots, u^m;\ \varphi^1, \dots \varphi^{n-m-1})$ give one of the sheets locally; then

$$r(u^1, \dots, u^m;\ \varphi^1, \dots, \varphi^{n-m-1}) \to r(u^1, \dots, u^m;\ \varphi^1, \dots, \varphi^{n-m-1};$$
$$t(u^1, \dots, u^m;\ \varphi^1, \dots, \varphi^{n-m-1})) = r'(u^1, \dots, u^m;\ \varphi^1, \dots, \varphi^{n-m-1})$$

Therefore

$$\text{(A)} \qquad \frac{\partial r'}{\partial u^1} = \frac{\partial r}{\partial u^1} + \frac{\partial r}{\partial t}\frac{\partial t}{\partial u^1} = \frac{\partial r}{\partial u^1} + \frac{\partial t}{\partial u^1}\dot{c}(t)$$

Similarly

$$\text{(B)} \qquad \begin{gathered} \frac{\partial r'}{\partial u^m} = \frac{\partial r}{\partial u^m} + \frac{\partial t}{\partial u^m}\dot{c}(t) \\ \frac{\partial r'}{\partial \varphi^1} = \frac{\partial r}{\partial \varphi^1} + \frac{\partial t}{\partial \varphi^1}\dot{c}(t) \\ \cdots\cdots\cdots\cdots \\ \frac{\partial r'}{\partial^{n-m-1}} = \frac{\partial r}{\partial \varphi^{n-m-1}} + \frac{\partial t}{\partial \varphi^{n-m-1}}\dot{c}(t). \end{gathered}$$

Here $\partial r/\partial u^1, \dots, \partial r/\partial u^m, \partial r/\partial \varphi^1, \dots, \partial r/\partial \varphi^{n-m-1}$ are Jacobi fields because they can be obtained from variations of c.

Assume now that M is of constant sectional curvature K. Let $E_1(t), \dots, E_{n-1}(t)$ be paralel orthonormal fields along c such that

1. $E_1(t), \dots, E_{n-1}(t)$ are orthogonal to $\dot{c}(t)$,

2. $E_1(0) = \partial r/\partial u^1 \ |_z \ \dots,\ E_m(0) = \partial r/\partial u^m \ |_z \in T_zL,\ E_{m+1}(0), \dots, E_{n-1}(0) \in N_zL$. Then taking $a_i = 1$ in

$$Y(t) = \sum_{i=1}^{m} a_i(\sin(\sqrt{K}t) - \sqrt{K}/\lambda_i \cdot \cos(\sqrt{K}t))E_i$$

we find

$$\partial r/\partial u^1 = \sin(\sqrt{K}t) - \sqrt{K}/\lambda_1 \cdot \cos(\sqrt{K}t))E_1(t)$$
$$\cdot$$
$$\cdot$$
$$\cdot$$
$$\partial r/\partial u^m = (\sin(\sqrt{K}t) - \sqrt{K}/\lambda_m \cdot \cos(\sqrt{K}t))E_m(t).$$

To find the values of $\partial r/\partial\varphi^1, \dots, \partial r/\partial\varphi^{n-m-1}$, we fix the point z like in the case where we considered a conjugate point. This gives $b_i = 0$ from

$$Y(t) = \sum_{i=1}^{n-1}(a_i \sin(\sqrt{K}t) + b_i \cos(\sqrt{K}t))E_i$$

so the values of $\partial r/\partial\varphi^1, \dots, \partial r/\partial\varphi^{n-m-1}$ are given by

$$\partial r/\partial\varphi^1 = \sin(\sqrt{K}t)E_{m+1}(t), \dots, \partial r/\partial\varphi^{n-m-1} = \sin(\sqrt{K}t)E_{n-1}(t).$$

Substituting these values of $\partial r/\partial u^1, \dots, \partial r/\partial u^m$, $\partial r/\partial\varphi^1, \dots, \partial r/\partial\varphi^{n-m-1}$, in (A) and (B) and then taking the wedge product of $\partial r'/\partial u^1, \dots, \partial r'/\partial u^m$, $\partial r'/\partial\varphi^1, \dots, \partial r'/\partial\varphi^{n-m-1}$, we have

$$\begin{aligned}
&\partial r'/\partial u^1 \wedge \cdots \wedge \partial r'/\partial u^m \wedge \partial r'/\partial\varphi^1 \wedge \cdots \wedge \partial r'/\partial\varphi^{n-m-1} = \\
&[(\sin\sqrt{K}t - \sqrt{K}/\lambda_1 \cdot \cos\sqrt{K}t)E_1(t) + \partial t/\partial u^1 \cdot \dot{c}(t)] \wedge \cdots \wedge \\
&\wedge [\sin(\sqrt{K}t)E_{m+1}(t) + \partial t/\partial\varphi^1 \cdot \dot{c}(t)] \wedge \cdots \wedge [\sin(\sqrt{K}t)E_{n-1}(t) + \\
&+ \partial t/\partial\varphi^{n-m-1} \cdot \dot{c}(t)] = (\sin\sqrt{K}t - \sqrt{K}/\lambda_1 \cdot \cos\sqrt{K}t) \cdots (\sin\sqrt{K}t - \sqrt{K}/\lambda_m \cdot \cos\sqrt{K}t) \\
&\sin^{n-m-1}(\sqrt{K}t)E_1(t) \wedge \cdots \wedge E_{n-1}(t) + \\
&\sum_{i=1}^{m}(\sin\sqrt{K}t - \sqrt{K}/\lambda_1 \cdot \cos\sqrt{K}t) \cdots (\sin\sqrt{K}t - \sqrt{K}/\lambda_i \cdot \cos\sqrt{K}t) \\
&\quad \partial t/\partial u^{i+1}(\sin\sqrt{K}t - \sqrt{K}/\lambda_{i+2}\cos\sqrt{K}t) \cdots (\sin\sqrt{K}t - \sqrt{K}/\lambda_m \cos\sqrt{K}t) \\
&\qquad \sin^{n-m-1}\sqrt{K}t E_1(t) \wedge \cdots \wedge E_i(t) \wedge \dot{c}(t) \wedge E_{i+2}(t) \wedge \\
&\qquad \wedge \cdots \wedge E_{n-1}(t) + \sum_{i=1}^{n-m-1}(\sin\sqrt{K}t - \sqrt{K}/\lambda_1 \cos\sqrt{K}t) \cdots \\
&\qquad \cdots (\sin\sqrt{K}t - \sqrt{K}/\lambda_m \cos\sqrt{K}t)\sin^{n-m-1}\sqrt{K}t\ \partial t/\partial\varphi^j \\
&\qquad E_1(t) \wedge \cdots \wedge E_m(t) \wedge E_{m+1}(t) \wedge \cdots \wedge E_{m+j-1}(t) \wedge \dot{c}(t) \wedge \\
&\qquad \wedge E_{m+j+1}(t) \wedge \cdots \wedge E_{n-1}(t).
\end{aligned}$$

Since $\dot{c}(t), E_1, \dots, E_{n-1}$ are n-linearly independent vectors, their wedge product is not zero. Let the point be first focal point; then right hand side of the above expression is not zero if

$$\begin{aligned}
\text{(i)}\quad &\partial t_1/\partial u^i(\sin\sqrt{K}t_1 - \sqrt{K}/\lambda_2\cos\sqrt{K}t_1) \cdots (\sin\sqrt{K}t_1 - \\
&-\sqrt{K}/\lambda_m \cos\sqrt{K}t_1)\sin^{n-m-1}\sqrt{K}t_1 \neq 0,\ \ i = 1, \dots, m \\
\text{(ii)}\quad &\partial t_1/\partial\varphi^j \neq 0,\ \ j = 1, \dots, n-m-1.
\end{aligned}$$

Thus we have

THEOREM. *Let M be a C^∞ n-dimensional complete Riemannian manifold of constant sectional curvature $K > 0$, and L be a C^∞ m-dimensional submanifold* *imbedded in M. Let* $c : [0,a] \to M$ *be a normal geodesic with* $c(0) = z \in L$, $\dot{c}(0) = v \in N_z L$. *Let also the position of focal points of* L *along* c *are given by*

$$1/\sqrt{K} \cdot \arctan(\sqrt{K}t) = \rho_i, \quad i = 1, \dots, m.$$

Then the conditions for the first sheet of focal points to be hypersurface are given by

(i) $\lambda_1 \neq \lambda_2, \lambda_3, \dots, \lambda_m$, (ii) $t_1 \neq 2\pi s/\sqrt{K}$, *s an integer,*

(iii) t_1 *is not independent of* $u^1, \dots, u^m$; $\varphi^1, \dots, \varphi^{n-m-1}$.

Analogous theorem may be stated when $K < 0$. The case when $a < n-1$ can be studied in a similar way.

The autor is thankful to Prof. J. Szente for providing useful discussions during the preparation of this paper.

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(Received 14 11 1983)