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SOME REMARKS ON M-CONVEXITY AND BEST APPROXIMATION

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Abstract. To study the uniqueness of best approximation properties for M-convex subsets of metric spaces, strictly M-convex and uniformly M-convex metric spaces were introduced in [**2**] by using the notion of M-convexity in metric spaces. In this note it is shown that strictly M-convex and uniformly M-convex metric spaces do not serve any fruitful purpose for the uniqueness of solutions of best approximation problems (the very purpose for which these spaces were introduced) as these prove the uniqueness of best approximation problems only when they are Mengerian; however, Mengerian spaces in the sense of [**2**] do not exist. We also answer some of the problems raised in [**2**] and show that some of the results proved in [**2**] are incorrect.

M-convexity for metric spaces was introduced in [**2**] as follows.

Definition 1. A metric space (X, d) is said to be *M-convex* if for every x, y in X, $x \neq y$, there exists a z in X different from x and y such that $d(x,y) = d(x,z) + d(z,y)$.

As remarked in [**2**], every normed linear space is an M-convex metrix space but not every M-convex metric space is a normed space. We may remark that every convex metric space is M-convex (a metric space (X, d) is said to be a convex metric space if the metric d is convex, i. e. if for each x, y in X, d determines at least one mid-point $z \in X$ in the sense $d(x,z) = d(z,y) = d(x,y)/2$) but an M-convex metric space need not be convex as the following example [**2**, Example 2.3] shows.

Let $G = \{(x,y) :\ 0 \leq x \leq 2,\ y = 0\} \cup \{(x,y) :\ x = 2,\ 0 \leq y \leq 1\} \subseteq R^2$ with $d((x_1,y_1),\ (x_2,y_2)) = \text{Max}\{\mid x_1 - x_2 \mid, \mid y_1 - y_1 \mid\}$. Then (G, d) is M-convex but not convex.

Moreover, an M-convex metric space need not even be a linear metric space [**2**, Example 1.4].

Using the notion of M-convexity in metric spaces, strictly M-convex and uniformly M-convex metric spaces were introduced in [**2**] as follows.

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Definition 2. A metric space (X,d) is said to be *strictly M-convex* if for every x,y,t in X, all different, and $r>0$, there exists a z in X different from x,y and t such that.

(1) $d(x,y)=d(x,z)+d(x,y)$

(2) $d(x,t)\le r,\ d(y,t)\le r$, imply $d(z,t)<r$.

Definition 3. A metric space (X,d) is said to be *uniformly M-convex* if to every pair of positive numbers ε and r, there corresponds a positive number δ such that every triplet x,y,t in X, all different, and satisfying $d(x,y)\ge\varepsilon$, $d(x,t)<r+\delta$, $d(y,t)<r+\delta$, there exists a z in X with the properties

(1) $d(x,y)=d(x,z)+d(z,y)$ (2) $d(z,t)<r$.

These two concepts are similar to those of strictly convex metric space and uniformly convex metric space introduced in [**1**]. As claimed in [**2**, Proposition 1.9] every uniformly M-convex metric space is strictly M-convex but not conversely; no example is given to support that strict M-convexity need not imply uniform M-convexity. Also, the example of uniformly M-convex space [**2**, Example 1.8] is not clear. Furthermore the proof of the following result [**2**, Theorem 1.11] relating strictly M-convex and uniformly M-convex spaces is defective.

THEOREM 1. *Every totally complete strictly M-convex metric space is uniformly M-convex.*

First, the construction of the set $S_t=\{(x,y)\in X\times X : d(x,t)\le r\}$ is defective it shoul be

$$S_t=\{(x,y)\in X\times X : d(x,t)\le r,\ d(y,t)\le r,\ d(x,y)\ge\varepsilon\}.$$

Second, the function $\Phi_t : S_t\to\mathbf{R}$ defined as $\Phi_t((x,y))=r-d(z,t)$, where $d(x,z)+d(z,y)=d(x,y)$, is not single-valued as there may be more than one z betwen x and y. So, unless the uniqueness of the point z is quaranteed (as in [**1**]), the continuity of Φ_t does not follow, and so Theorem 1 may not hold.

M-convex subsets of a metric space were introduced in [**2**] as follows.

Definition 4. A subset G of a metric space (X,d) is aid to be *M-convex* if for every x,y in $G, x\ne y$, there exists a z in G such that $d(x,z)+d(z,y)=d(x,y)$.

It is clear that this definition is meaningless unless it is required that z is different from x and y, otherwise every non-empty set will be M-convex. So we shall assume that it is a part of Definition 4.

it was remarked in [**2**] that there is no relation between convexity and M-convexity in a metric linear space. However as it is easy to see, one can say the following:

Every convex set (cf. [**1**]) in an M- convex metric space is M-convex.

An example of a metric linear space in which there is an M-convex set which is not convex is given in [**2**, Example 2.3], and it is remarked in [**2**] that in a normed linear space an M-convex set is always convex. However, this is not true, as the same example (Example 2.3) shows.

It is remarked in [**2**] that in general, proximinal sets, or Chebyshev sets, are neither convex not M-convex. Also the following question is raised in [**2**]:

Whether in a Hilbert space, every Chebyshev set is M-convex? The answer is yes. In fact, we can say much more.

THEOREM 2. *Any closed set in a normed linear space is M-convex.*

Proof. Let G be a closed set in a normed linear space X. Suppose G is not M-convex. This means that there exists $x, y \in G$, $x \neq y$, such that for these x and y there is no z in G, different from x and y such that

$$d(x,z) + d(z,y) = d(x,y) \tag{1}$$

i. e. any z satisfying (1) does not lie in G. Consider

$$\begin{aligned}\operatorname{dist}((x+y)/2, G) &= \inf\{\|(x+y)/2 - g\| : g \in G\} = 1/2 \inf\{\|(x-g)+(y-g)\| g \in G\} \\ &\leq 1/2 \inf\{\|x-g\| + \|y-g\| : g \in G\}.\end{aligned}$$

This gives, $\operatorname{dist}((x+y)/2,\ G) \leq 1/2 \operatorname{dist}\,(x,G) + 1/2 \operatorname{dist}\,(y,G) = 0$ i. e.

$$\operatorname{dist}((x+y)/2,\, G) = 0 \text{ i.e. } (x+y)/2 \in \overline{G} = G \text{ and also}$$
$$\|x-(x+y)/2\| + \|y-(x+y)/2\| = \|x-y\|/2 + \|x-y\|/2 = \|x-y\| \text{ i.e. } (x+y)/2$$

lies between x and y, a contradiction.

Remark. This shows that every proximinal (Chebyshev) set in any normed linear space is M-convex. The question whether it is convex is still open. The answer is not known even in Hilbert spaces (cf. [**3**]).

Menger sets were defined in [**2**] as follows.

Definition 5. In a metric space (X,d), a *Menger set*, denoted as $M_{\langle x,y\rangle}$ for a pair of distinct points x, y is defined as the set of elements z in X such that $d(x,y) + d(z,y) = d(x,y)$ i.e. $M_{\langle x,y\rangle} = \{z \in X : d(x,z) + d(z,y) = d(x,y)\}$.

It was remarked in [**2**] that Menger sets can be empty sets, singelton sets or arbitrary large set. It is clear that $x, y \in M_{\langle x,y\rangle}$, and so, unless one imposes the condition that z is different from x and y, Menger sets can never be empty or singleton. But if one imposes this condition, then Proposition 2.9, which asserts that the Mengen sets are always closed, is false as Menger sets can be open; e. g. on the real line $\mathbf{R}$ with usual norm, the set $]x, y[$ is a Menger set. Moreover, if we do not impose the above condition, then no metric space is Mengerian. Mengerian metric spaces were defined in [**2**] as:

Definition 6. If a metric space has only singleton Menger sets for every pair of distinct elements, then it will be called *Mengerian*.

Furthermore, if Mengerian spaces do not exist, then Theorem 2.11 and Theorem 2.12 do not have any significance (there are also some misprints in the proofs of these two theorems).

The definition of approximatively compact set (Definition 2.12) is defective. It should be (cf. [**4**]):

Definition 7. A set G in a metric space (X, d) is said to be *approximatively compact* if for every $x \in X$ and every sequence $\langle y_n \rangle$ in G with $\lim_n d(x, y_n) = d(x, G)$, there exists a subsequence $\langle y_{n_k} \rangle$ converging to an element of G.

It is required in the proof of Theorem 2.13 [**2**] that Definition 7 holds for every $x \in X$.

Metric spaces with property-P (Definition 2.14) were introduced in [**2**] as follows.

Definition 8. A metric space (X, d) is said to have the *P-property* if for a fixed p in X, every sequence $\langle y_n \rangle$ in an M-convex set G of X satisfying $\lim_n d(p, y_n) = d(p, G)$ has a Cauchy subsequence.

For metric spaces with P-property the following result (Theorem 2.15) was proved in [**2**].

THEOREM 3 *A complete M-convex subset G of a metric space (X, d) having the P-property is Chebyshev.*

First, there is no need of taking the set G to be M-convex in Definition 8 as well as in Theorem 3. Second, if we only require the existence of a Cauchy subsequence in Definition 8, then the proof of the uniqueness part of Theorem 3, as given in [**2**], is incorrest, for by the P-property, $\langle y_n \rangle$ has a Cauchy subsequence (it may be $\langle y_{2k} \rangle$ or $\langle y_{2k+1} \rangle$). From this it does not follow that $y_1 = y_2$, since y_1 or y_2 need not at all appear in the Cauchy subsequence.

The proof of Theorem 3, as given in [**2**], may work if we require in the definition that $\langle y_n \rangle$ itself is a Cauchy sequence. Moreover, it will be better if Definition 8 is given for sets (see Definition 4 of [**1**] for a definition of approximatively Cauchy sets).

It may also be noted that the proof of Theorem 2.16 (Every uniformly M-convex Mengerian metric space has the P-property) is embodied in Theorem 2.13, rather than in Theorem 2.14 as noted in [**2**]. Moreover, this result will be valid even with an improved form of Definition 8, in which we require that $\langle y_n \rangle$ itself is a Cauchy sequence.

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