PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 37 (51), 1985, pp. 61–63

INEQUALITIES FOR ELLIPTIC INTEGRALS

G. D. Anderson , M. K. Vamanamurthy

Abstract. Sharp lower and upper estimates are obtained for five expressions involving complete elliptic integrals. The proofs for four of these are elementary, while the last involves contour integration and some inequalities for elliptic functions.

Several authors, noting the apparent paucity of inequalities involving elliptic functions and elliptic integrals, have presented results of this nature ([**1, 2, 5, 7**]). In this note we offer several inequalities for elliptic integrals that may be obtained elementary methods. Such inequalities are of interest in part because of the connestion of complete elliptic integrals of the first kind with the moduls of the Grötzsch extremal ring in the plane and the theory of plane quasiconformal mapping (cf. [**8**]).

We show that the following sharp inequalities hold for $0 < k < 1$;

$$\pi k^2/4 < E - k'^2 K < k^2, \tag{1}$$

$$\log 4 < K + \log k' < \pi/2, \tag{2}$$

$$1 < \frac{K}{\log(4/k')} < \frac{\pi}{2\log 4}, \tag{3}$$

$$1 < k^2 \exp(\pi K'/K) < 16, \tag{4}$$

$$\frac{4k'^2 K^2}{\pi^2} < \tanh\frac{\pi K'}{2K} < \frac{16-k^2}{16+k^2}, \tag{5}$$

where K and E are the complete elliptic integrals of the first and second kinds, respectively,

$$K = \int_0^{\pi/2} (1 - k^2 \sin^2 t)^{-1/2} \delta t, \quad E = \int_0^{\pi/2} (1 - k^2 \sin^2 t)^{1/2} \delta t \tag{6}$$

([**3**, p. 17], [**4**, #110.06-.07]) and $K' = K(k')$, $k' = (1-k^2)^{1/2}$.

AMS Subject Classification (1980): Primary 33 A 25.

To prove (1) we let

$$f(k) = \frac{E - k'^2 K}{k^2} = \int_0^{\pi/2} (1 - k^2 \sin^2 t)^{-1/2} \cos^2 t \delta t$$

for $0 < k < 1$, and put $f(0) = \pi/4$ and $f(1) = 1$. Then clearly f is strictly increasing on $[0, 1]$, and inequality (1) follows.

For (2) we obsrve first that

$$\delta/\delta k(K + \log\, k') = (E - k'^2 K - k^2)/kk'^2 < 0$$

by (1) and by [**3**, p. 21] or [4, # 710.00], while

$$\lim_{k \to 1}(K + \log\, k') = \log\, 4, \quad \lim_{k \to 0}(K + \log\, k') = \pi/2$$

by [**3**, p. 21] or [4, # 112.01] and by (6).

Next consider (3). We see that

$$(\log(4/k'))^2 \frac{\delta}{\delta k}\left(\frac{K}{\log(4/k')}\right) = \left[\left(\log\frac{4}{k'}\right)\frac{E - k'^2 K}{kk'^2} - \frac{kK}{k'^2}\right],$$

which is negative if and only if

$$\log(4/k') < k^2 K/(E - k'^2 K) \tag{7}$$

since $E - k'^2 K > 0$ by (1). But (7) follows from the first half of (2) and the second half of (1). The sharp bounds in (3) now follow from the limit $\lim\limits_{k \to 1} K/(\log(4/k')) = 1$ ([**3**, p. 21], [4, #112.01]) and by (6).

For (4) we note that

$$\frac{\delta}{\delta k}(k^2 e^{\pi K'/K}) = \frac{ke^{\pi K'/K}}{2k'^2 K^2}(-\pi^2 + 4k'^2 K^2)$$

(cf. [**3, 4**], or [**6**]), which is negative by (2) of [**1**]. The sharp bounds in (4) now follow from [**3**, p. 22] or [4, #112.04] and the obvious limit as k tends to 1.

The bounds

$$\frac{1 - k^2}{1 + k^2} < \tanh\frac{\pi K'}{2K} < \frac{16 - k^2}{16 + k^2} \tag{8}$$

follow easily from (4) and the fact that $(t - 1)/(t + 1)$ is an increasing function on $[1, \infty)$. However, we may derive the better lower bound in (5) by performing the contour integration

$$\int_C e^{\pi i z/K} \mathrm{cn}(z, k)\mathrm{dn}(z, k)/\mathrm{sn}(z, k)\delta z,$$

where C is the rectangle with vertices at $\pm K$, $\pm K + 2iK'$ but with semicircular indentations of radius ε at 0 and $2iK'$. Using the fact that the integrand has a simple pole at iK' with residue $-\exp(-\pi K'/K)$ [**4**, pp. 18–19] and afterwards letting ε tend to zero, one may show that

$$\tanh\frac{\pi K'}{2K}=\frac{2}{\pi}\int_0^K \sin\frac{\pi x}{K}\operatorname{cn}(x,k)\operatorname{dn}(x,k)/\operatorname{sn}(x,k)\delta x$$

(cf. [**3**, p. 43, Example 8], [**9**, p. 532, Example 39]). The change of variable $t=\pi x/2K$ in the integral reduces this equation to

$$\tanh\frac{\pi K'}{2K}=\frac{4K}{\pi^2}\int_0^{\pi/2}\frac{cd}{s}\sin\,2t\,\delta t, \tag{9}$$

where $s=\operatorname{sn}(2Kt/\pi,k)$, $c=\operatorname{cn}(2Kt/\pi,k)$, $d=\operatorname{dn}(2Kt/\pi,k)$. But

$$cd/s\geq 2k'^2K\pi^{-1}\cot\,t \tag{10}$$

by (4) of [**1**] and the fact that $d\geq k'$. Combining (9) and (10) we now obtain the first inequality in (5), which is sharp because of [**3**, p. 21] or [4, # 112.01].

REFERENCES

[1] G. D. Anderson, *Inequalities for elliptic functions*, Amer. Math. Monthly **74** (1967), 1072–1074.

[2] G. D. Anderson, M. K. Vamanamurthy, *Affine mappings and elliptic functions*, Punb. Inst. Math. (Beograd) (N. S) **11(25)** (1971), 49–51.

[3] F. Bowman, *Introduction to Elliptic Functions with Applications*, Dover, New York, 1961.

[4] P. F. Byrd, M. D. Friedman, *Handbook of Elliptic Integrals for Engineers and Physicists*, Springer-Verlag, New York, 1954.

[5] P. L. Duren, *Two inequalities involving elliptic functions*, Amer. Math. Monthly **70** (1963), 650–651.

[6] A. Enneper, *Elliptische Functionen, Theorie und Geschichte*, Louis Nebert, Halle a. S., 1890.

[7] S. Fempl, *Some inequalities involving elliptic functions*, Amer. Math. Monthly **72** (1965), 150–152.

[8] O. Lehto, K. I. Virtanen, *Quasiconformal Mappings in the Plane*, Second Ed., Springer-Verlag, New York, 1973.

[9] E. T. Whittaker, G. N. Watson, *A Course of Modern Analysis*, Fourth Ed., Cambridge Press, 1958.

Michogan State University
East Lansing Michigan, USA
48824

(Received 07 08 1984)

University of Auckland
Auckland, New Zealand