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ALGORITHMIC PROBLEMS RELATED TO THE DIRECT PRODUCT OF GROUPS

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Abstract. Algorithmic unrecognizability of every group property from classes $\mathcal{K}_1'$ and $\mathcal{K}_2'$ of properties (of fp groups) related to the direct product of groups, is proved. $\mathcal{K}_1'$ is the class of all properties of the form "being a direct product of groups with Markov properties". $\mathcal{K}_2$ is the class of properties $P = S \cup T$, where S is a property of universal fp groups only and T is a property of nonuniversal groups, such that there exists a positive integer m for which

$$(\forall\, G \in S) d(G) \neq m \vee (\forall\, G \in T) d(G) \neq m,$$

where $d(G) = \sup\{k \mid (\exists\, H_1, \dots, H_k \neq \{1\})(G \cong H_1 \times \cdots \times H_k)\}$.

Introduction. Algorithmic recognizability of group properties, involving the notion of the free product of groups, was frequently considered in the literature. Let us first give some example of such properties. Let $\mathcal{F}$ denote the set of all finitely presentable (fp) groups, and let *a property* mean a property of fp groups which is algebraic, i. e. which is preserved under any isomorphism. Let further *a universal group* stand for an fp group which contains (as a subgroup) an isomorphic copy of every fp group.

Example 1. Let $\mathcal{G}_1$ be the class of all the properties defined by

$$F_p(G) \Leftrightarrow (\exists\, n \in N)(\exists\, G_1, \dots, G_n \in \mathcal{F})(G \cong G_1 * \cdots * G_n \wedge (\forall\, i \leq n) P(G_i)).$$

Thus a group G enjoys the property F_p if it is a free product of groups which enjoy the property P.

Example 2. Let $f(G)$ denote the maximal number of nontrivial factors in a free decomposition og G, i. e. let

$$f(G) = k \Leftrightarrow (\exists\, G_1, \dots, G_k \neq \{1\}) G \cong G_1 * \cdots * G_k \wedge (\forall\, m \in N)((\exists\, H_1, \dots, H_m \neq \{1\}) G \cong H_1 * \cdots * H_m \Rightarrow m \leq k).$$

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For every fp group, the number $f(G)$ is a uniquely defined positive integer. Let now $\mathcal{G}_2$ denote the class of all the properties of the form $Q \cup R$, where Q is a property of universal fp groups only, and R is a property of nonuniversal groups, such that

$$(\exists\, k \in N)(\forall\, G \in R) f(G) \neq k \vee (\exists\, m \in N)(\forall\, H \in Q) f(H) \neq m.$$

Some examples of such properties are the following ones: being decomposable into a free product of universal groups ($m = 1$), having a nontrivial centre ($k \geq 2, m \geq 2$), etc.

In te sequel we consider the problem of algorithmic (i. e. recursive) recognizability of properties of fp groups, i.e. the problem of existence of an algorithm which for a given presentation Π recognizes whether the group G_Π determined by that presentation enjoys the considered property, or not.

As far as the classes of properties defined above are concerned, the following results are valid.

STATEMENT 1. [**4**] *Let $\mathcal{G}_1'$ be the subclass of all the properties F_p from $\mathcal{G}_1$, for which P is a Markov property. Then every property from $\mathcal{G}_1'$ is unrecognizable.*

Remark. For one particular property belonging to this class (being a free product of finite groups), unrecognizability was proved earlier by Rabin [**8**].

STATEMENT 2. *Every property from the class $\mathcal{G}_2$ is unrecognizable.*

A related result was proved in [**5**]: the class $\mathcal{G}_2'$ defined by

$$(\exists\, k \in N)(\forall\, G \in R)((f(G) \neq k \wedge \min_{G \in R} f(G) > 1) \vee (\exists\, m \in N)(\forall\, H \in Q) f(H) \neq m$$

contains only unrecognizable properties. This result can be extended now onto the whole class $\mathcal{G}_2$. Namely, for the group G_r, which appears in the proof on p. 67 in [**5**], one can choose $G_r = N \times G_{\Pi_{(r)}}$, where N is a nonuniversal group which enjoys the property P. From Lemma 2 below, it follows that G_r is not a universal group; then $P(G_r) \Leftrightarrow G_{\Pi_{(r)}} \cong \{1\}$, etc. as in [**5**].

Utilizing some other group constructions instead of the free product , one can define classes of properties analogous to the classes $\mathcal{G}_1$ and $\mathcal{G}_2$ defined above. In this paper, the direct product of groups is considered, and it is shown that Statements 1 and 2 remain valid if one replace "the free product" by "the direct product".

For certain classes of properties, such an analogy is easily seen. For example, let $\mathcal{G}_3$ denote the class of all the properties P *incompatible with the free product*, i.e. the properties for which

$$(\exists\, G)(\forall\, H \neq \{1\})(P(G) \wedge \neg P(G * H)).$$

Every property from the class $\mathcal{G}_3$ is unrecognizable [**7**, p. 193].

Now, let $\mathcal{K}_3$ denote the class of all the properties *incompatible with the direct product*, i.e. such that

$$(\exists\, G)(\forall\, H \neq \{1\})(P(G) \wedge \neg P(G \times H).$$

Every property from the class $\mathcal{K}_3$ is also unrecognizable and the proof is completely analogous.

However, the transfer is not always so straightforward and some careful consideration may be necessary; that is the case for the classes $\mathcal{K}_1$ and K_2 defined analogously to the classes $\mathcal{G}_1$ and $\mathcal{G}_2$.

The following terminology is utilized in the sequel. A group G is the *direct product* of its subgroups G_i (i ranges over a given index set I) if the elements from any two distinct subgroups are permutable, and if every element $g \in G$ has a unique representation as a product of finite number of elements chosen from the subgroups G_i.

Nontrivial subgroups G_i are (direct) *factors* of group G. A group G is *indecomposable* (into the direct product) if

$$(\forall G_1)(\forall G_2)(G \cong G_1 \times G_2 \Rightarrow (G_1 \cong \{1\} \vee G_2 \cong \{1\})).$$

The decomposition $G = \Pi_{i,j} G_{ij}$ is a *refinement* of the decomposition $G = \Pi_i G_i$, if $G_i = \Pi_j G_{ij}$.

Statements and proofs. In the sequel, essential use is made of the following two lemmas.

LEMMA 1. [**4**] *A universal fp group cannot be decomposed into a free product of nonuniversal fp groups.* □

LEMMA 2. *A universal fp group cannot be decomposed into a direct product of two nonuniversal fp groups.*

Proof. Let G_1 and G_2 be nonuniversal groups and G be a group which is not embeddable in either G_1 or G_2. We prove that $G * G$ is not embeddable in $G_1 \times G_2$. Assume, on the contrary, that the free product $P * Q$, with $P, Q \cong G$ is a subgroup of $G_1 \times G_2$. Let $H_i (i = 1, 2)$ be the image of $P * Q$ under the projection $G_1 \times G_2 \to G_i$ and let $A_i = (P * Q) \cap G_i$. Thus, $P * Q$ is a subdirect product of H_1 and H_2 and $(P * Q)/A_1 \cong H_2$ and $(P * Q)/A_2 \cong H_1$.

The direct product $A_1 \times A_2$ is a normal subgroup of $P * Q$ and, by Kurosh's Subgroup Theorem, it decomposes as $A_1 \times A_2 = F * \Pi_i^* B_i$ where F is a free group and B_i are conjugates of subgroups of P or Q in $P * Q$. Here A_1 and A_2 are nontrivial groups (otherwise $P * Q \cong H_1$ or H_2 and G is embeddable in G_1 or G_2) and we invoke the fact that a nontrivial direct product is never a nontrivial free product (see for example [**7**, p. 177]). Therefore, since free groups are not decomposable into a nontrivial direct product, it follows that $A_1 \times A_2$ is a conjugate of a subgroup of P or Q. Hence $A_1 \times A_2$ cannot be normal in $P * Q$, contrary to the above assumption. □

For a given property P, let D_p denote the property of "being a direct product of groups with the property P", i. e. let

$$D_P(G) \Leftrightarrow (\exists n \in N)(\exists G_1, \dots, G_n \in \mathcal{F})(G \cong G_1 \times \cdots \times G_n \wedge (\forall i \leq n) P(G_i)).$$

Let further $\mathcal{K}_1'$ denote the class of all properties D_P for which P is a Markov property. Then the following statement is true.

THEOREM 1. *Every property from the class $\mathcal{K}_1'$ is recursively unrecognisable.*

Proof. Let G be an arbitrary group enjoying a property D_P, where P is a given Markov property. Then $G \cong G_1 \times \cdots \times G_k$, where the groups $G_i (i = 1, \ldots, k)$ enjoy the property P. Since P is a Markov property, none of the groups G_i is universal [**3**]. In vew of Lemma 2 G is also a nonuniversal group, and thus no universal group can enjoy the property D_P. Therefore, D_P is a Markov property, and hence unrecognizable [**8**]. $\square$

Remark. In the same way one can prove that every property D of the form

$$D(G) \Leftrightarrow (\exists n \in N)(\exists G_1, \ldots, G_n \in \mathcal{F})(G \cong G_1 \times \cdots \times G_n \wedge)(\forall i \leq n) P_i(G_i))$$

where $P_1, P_2, \ldots, P_n$ are arbitrary Markov properties, is unrecognizable. Similarly, Statement 1 can be slightly strenghtened: any property F defined by

$$F(G) \Leftrightarrow (\exists n \in N)(\exists G_1, \ldots, G_n \in \mathcal{F})(G \cong G_1 * \cdots * G_n \wedge (\forall i \leq n) P_i(G_i))$$

where $P_1, P_2, \ldots, P_n$ are arbitrary Markov properties, is unrecognizable.

Let us now consider the number of direct factors of fp groups, in order to prove the direct product analogue of the Statement 2. As far as decomposability is concerned, the direct product is more difficult to deal with than the free product. For example, if $G \cong G_1 * \cdots * G_n$ and $G \cong H_1 * \cdots * H_m$ are two decompositions of the group G such that neither of the free factors is further decomposable into a nontrivial free product, then $n = m$ and $(\forall i)(\exists j) G_i \cong H_j$, where $i, j \leq n$. However, there is not such a general decomposition theorem for direct product: there exists a group G such that $G \cong A \times B$ and $G \cong C \times D$ are two nonisomorphic direct decompositions of G, and where none of A, B, C and D is decomposable into a direct product (see e.g. [**6**, vol. II p. 81]). But, if we consider the class of finitely generated groups, then the following is true.

LEMMA 3. *A finitely generated group is not decomposable into an infinite direct product of nontrivial groups.*

Proof. Let G be an infinite direct product of its nontrivial subgoups G_i i.e. $G = \Pi_{i=1}^{\infty} G_i$, and let $a_1, a_2, \ldots, a_n$ be generators for G. Let $F_j = \Pi_{i=1}^{j} G_i$; $F_1 \leq F_2 \leq \cdots \leq F_n \leq \cdots$ is an ascending series of subgoups of the group G. Let g be an element of G; then g is a uniquely defined product of a finite number of elements of subgoups G_i, i.e. there exists an integer j such that $g \in F_j$, $F_j = G_1 \times G_2 \times \cdots \times G_j$. Hence, $G = \cup_{j=1}^{\infty} F_j$. On the other hand every F_j is a proper subgroup of the group G, i.e.

$$(\forall i) F_i < G \tag{1}$$

In particular, for every generator $a_i (i = 1, \ldots, n)$ there is group F_{j_i} such that $a_i \in F_{j_i}$. Let $k = \max\{j_1, \ldots, j_n\}$; then $(\forall i \leq n) a \in F_k$, i. e.

$$G \leq F_k. \tag{2}$$

But (2) contradicts (1), and hence the lemma is proved. □

Hence, every fp group G has a finite number of nontrivial indecomposable direct factors. Let us define the function

$$d(G) = \sup\{k \mid (\exists\, H_1, \ldots, H_k \neq \{1\})(G \cong H_1 \times \cdots \times H_k)\}.$$

Denoting the class of all universal fp groups by $\mathcal{U}$, and the class of all nonuniversal fp groups by $\mathcal{N}$, we have

THEOREM 2. *Let P be an algebraic property of fp groups such that $P = S \cup T$, where $S \subseteq \mathcal{N}$ and $T \subseteq \mathcal{U}$. If there exists a positive integer m such that*

$$(i)\ (\forall\, G \in S) d(G) \neq m, \quad \text{or} \quad (ii)\ (\forall\, G \in T) d(G) \neq m,$$

then P is a recursively unrecognizable property.

Proof. If $T = \varnothing$ the property P is a Markov property [**3**], and hence it is unrecognizable. Also, if $T = \mathcal{U}$, the complement of P in $\mathcal{F}$ is a Markov property. So in what follows we assume that $T \neq \varnothing$ and $T \neq \mathcal{U}$.

Denote by C_k the class of groups having k nontrivial direct factors but no more, i.e. let $C_k = \{G \mid G \in \mathcal{F} \wedge d(G) = k\}$. Each class $C_k (k \in N)$. contains a nonuniversal group without center and a universal group without center. Namely, let G ne an fp group and let $\overline{G}$ be defined by

$$\overline{G} = (G * G) \times \cdots \times (G * G) \qquad (k \text{ times}).$$

Evidently, $d(\overline{G}) \geq k$. A theorem of Kurosh [**6**, vol. II, p. 105] ensures that all decompositions of a given group have common refinement, i.e. that a group has a unique decomposition into indecomposable factors, if the group considered has no center. But $\overline{G}$ is a such group, since $G * G$ has no center, and the center of a direct product is the direct product of the centers of the factors. Further, $G * G$ is not decomposable into a direct product. Therefore $d(\overline{G}) = k$, i.e. $\overline{G} \in C_k$. In the above consideration G is an arbitrary fp group; if we choose it now to be a nonuniversal group, then $\overline{G}$ is also a nonuniversal fp group, as follows from Lemmas 1, and 2. Similarly, if we choose G to be a universal group, then $\overline{G}$ is also universal fp group, and thus the assertion is proved.

(i) Now let m be the smallest integer such that the property S contains none of the groups belonging to C_m. If $m > 1$, each of the classes $C_1, \ldots, C_{m-1}$ contains some groups from S. Let S_1 denote this subset of S, i.e. let

$$S_1 = \{G \mid (\exists\, i \in m-1) G \in C_i \wedge S(G)\}.$$

Then either (a) the set S_1 contains a group H without center, or (b) it contains only groups with nontrivial centers.

In the case (a) the group H can be uniquely decomposed into indecomposable factors: $H = H_1 \times H_2 \times \cdots \times H_i$ where $i = d(H) < m$. On the other hand, let $\Pi(r)$ be the presentation effectively construced from the pair (Π, r) by the Rabin's construction [**8**]; here r is a word from the presentation Π. The group $G_{\Pi_{(r)}}$ is isomorphic to $\{1\}$ if $\vdash_\Pi r = 1$, and it contains the group G_Π as a subgroup, otherwise. We are looking for such a finite presentation Π for which G_Π is a nonuniversal group with unsolvable word prolem, for which $G_{\Pi_{(r)}}$ is also a nonuniversal group. One possibility is to look for Π among the presentations of torsion-free groups, because there exists a torsion-free group G_Π with unsolvable world problem [**1**]; in that case $G_{\Pi_{(r)}}$ is also a torsion-free group [**2**] and hence it is not universal either. Let us now construct the group

$$G_r = H \times (G_{\Pi_{(r)}} * G_{\Pi_{(r)}}) \times \cdots \times (G_{\Pi_{(r)}} * G_{\Pi_{(r)}}) \quad (m-i \text{ times}).$$

If $\vdash_\Pi r = 1$, then $G_r \cong H$, so that $S(G_r)$ and thus $P(G_r)$. On the contrary, if $\vdash_\Pi r = 1$ is not true, then in view of Lemmas 1 and 2, G_r is a nonuniversal group, and it has no center. In addition, it can be decomposed into indecomposable factors as follows:

$$G_r = H_1 \times H_2 \times \cdots \times H_i \times (G_{\Pi_{(r)}} * G_{\Pi_{(r)}}) \times \cdots \times (G_{\Pi_{(r)}} * G_{\Pi_{(r)}}) \quad (m-i \text{ times}).$$

so that $d(G_r) = m$, and therefore $\neg P(G_r)$.

In the case (b), none of the groups without center that belongs to the classes $C_1, C_2, \cdots, C_{m-1}$ is contained in S_1, and hence none them ejoys the property P. Let K be an arbitrary group such that $S(K)$ is true; in that case we can choose the group G_r as follows:

$$G_r = K * G_{\Pi_{(r)}} \tag{3}$$

which is either isomorphic to K so that $P(G_r)$ is true, or $d(G_r) = 1$, i.e. G_r has no center in which case $P(G_r)$ is not true. If $m = 1$, G_r can be chosen as in (3).

(ii) Quite analogously one can deal with the case when the property T contains none of the universal groups from class C_m. In the case (a), i.e. when $T_1 = \{G \mid (\exists i \leq m-1) G \in C_i \wedge T(G)\}$, contains a universal group U without center, and if $m > 1$ one can choose G_r as follows

$$G_r = U \times (G_{\Pi_{(r)}} * G_{\Pi_{(r)}}) \times \cdots \times (G_{\Pi_{(r)}} * G_{\Pi_{(r)}}) \qquad (m-l \text{ times})$$

where $l = d(U) < m$. In the case (b), i.e. if T_1 contains only groups with nontrivial canters, then

$$G_r = U' * G_{\Pi_{(r)}} \tag{4}$$

where U' is a universal group which enjoys the property P. If $m = 1$, G_r can be chosen as in (4).

Let us remark that in the latter case, for $G_{\Pi_{(r)}}$ we can choose the group construced from arbitrary (universal or nonuniversal) group with unsolvable word problem.

In all the above cases, $P(G_r)$ iff $\vdash_\Pi r = 1$ and since the presentation Π is chosen to have an unsolvable word problem, the property P is not recognizable. $\square$.

Comments. Investigating systematically the problem of algorithmic recognizability of properties of groups, cetrain conditions which every reconizable property must satisfy were given in [**5**]. Now we can stregthen these conditions as follows:

If an algebraic property P of fp groups is recognizable, then

(1) P contains some but not all universal groups;

(2) for every positive integer k, P contains a nonuniversal group G_1 and a universal group U_1 such that $f(G_1) = f(U_1) = k$;

(3) for every positive integer k, P contains a nonuniversal group G_2 and a universal group U_2 such that $d(G_2) = d(U_2) = k$;

(4) for every recursively enumerable degree of unsolvability d, P contains a group G such that the degree of unsolvability of the word problem of G is of degree d or higher;

(5) the complement of P in $\mathcal{F}$ also satisfies the conditions (1)—(4).

These conditions might look rather restrictive, but still there are some simple properties which fulfill them all. For example, such are the property "having an Abelian quotient" and "having a quotient with exactly n elements". On the other hand, some properties might fulfill all of the above conditions (1)—(5) and still be unrecognizable.

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