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ON SOME GRAPHIC POLYNOMIALS WHOSE ZEROS ARE REAL

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Abstract. Polynomials which are formed by linear combination of the characteristic polynomial of a graph G and the characteristic polynomials of the vertex-deleted subgraphs of G have real zeros. The same is true for the linear combination of the matching polynomial of G and the matching polynomials of the vertex-deleted subgraphs of G. Several statements about the location of the zeros of these polynomials are obtained.

1. Introduction. Let G be a graph having n vertices, $n \geq 2$. Let the vertices of G be labelled by $v_1, v_2, \dots, v_n$. The subgraph obtained from G by deletion of v_r will be denoted by G_r.

Two polynomials associated with a graph have been extesively studied in the mathematical literature, namely the characteristic [**1**] and the matching polynomial [**2**]. They will be denoted by $\varphi(G)$ and $\alpha(G)$, respectively.

Both $\varphi(G)$ and $\alpha(G)$ are polynomials of degree n in the variable x. Their zeros will be denoted by $x_i, 1, 2, \dots, n$ and y_i, $i = 1, 2, \dots, n$, respectively. It is known [**1, 2**] that all x_i's and y_i's are real and that, in addition, the following interlacing relations holds:

(1) $x_i \leq x_i^r \leq x_{i+1}$ for $i = 1, \dots, n-1$

(2) $y \leq y_i^r \leq y_{i+1}$ for $i = 1, \dots, n-1$,

where x_i^r and y_i^r are the zeros of $\varphi(G_r)$ and $\alpha(G_r)$, respectively.

It is also known that

(3) $d\varphi(G)/dx = \sum_{r=1}^n \varphi(G_r)$, (4) $d\alpha(G)/dx = \sum_{r=1}^n \alpha(G_r)$.

We shall examine several classes of graphic polynomials and determine certain properties of their zeros.

Let A be an ordered n-tuple $(A_1, A_2, \dots, A_n)$ of positive real numbers.

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Let $B \subset \{1,2,\dots,n\}$. For given A, B and graph G we define following six polynomials:

$$\varphi^*(G) = \sum_{r\in B} A^r \varphi(G_r), \qquad \varphi^-(G) = \varphi(G) - \varphi^*(G)$$
$$\varphi^+(G) = \varphi(G) + \varphi^*(G),$$
$$\alpha^*(G) = \sum_{r\in B} A_r \alpha(G_r), \qquad \alpha^-(G) = \alpha(G) - \alpha^*(G),$$
$$\alpha^+(G) = \alpha(G) + \alpha^*(G).$$

Note that for $A_1 = A_2 = \cdots = A_n = 1$ and $B = \{1,2,\dots,n\}$, $\varphi^*(G)$ and $\alpha^*(G)$ are equal to the first derivatives of $\varphi(G)$ and $\alpha(G)$, respectively, eqs. (3) and (4)

The following theorem can be understood as the main result of the present work.

THEOREM 1. (a) *For all A, B and G, all the zeros of $\varphi^*(G)$, $\varphi^-(G)$, $\varphi^+(G)$, $\alpha^*(G)$, $\alpha^-(G)$ and $\alpha^+(G)$ are real.* (b) *If these zeros are denoted by x_i^*, x_i^- x_i^+ y_i^*, y_i^- and y_i^+, respectively, then for $i = 1,\dots,n-1$, $x_i^+ \le x_i \le x_i^- \le x_i^* \le x_{i+1}^+ \le x_{i+1} \le x_{i+1}^-$.*

2. Preliminaries. All the polynomials considered in the present paper will be assumed to have real coefficients and a positive leading coefficient, ($\varphi(G)$, $\alpha(G)$ and the polynomials introduced in Definition 1 meet, of course, these requirements.) The variable in all the polynomials considered is denoted by x.

Let P and Q be two polynomials of degree m and n, respectively. Let their zeros be $p_1, p_2, \dots, p_m$ and $q_1, q_2, \dots, q_n$, respectively.

We say that P *separates* Q in the following two cases;

(a) if $m = n-1$ and $q_i \le p_i \le q_{i+1}$ for $i = 1,\dots,n-1$, and

(b) if $m = n$ and $q_i \le p_i \le q_{i+1}$ for $i = 1,\dots,n-1$ and $q_n \le p_n$.

Then we shall write P sep Q.

The relation P sep Q implies, of course, that all the zeros of both P and Q are real. We shall need the following simple property of the separation relation.

LEMMA 1. *If a polynomial S exists, which separates the polynomials P, Q and R, then from P sep Q and Q sep R it follows that P sep R.*

Using the notation of Definition 2, we can formulate the inequalities (1) and (2) in the following manner.

LEMMA 2. *For all $r = 1,2,\dots,n$, $\varphi(G_r)$ sep $\varphi(G)$ and $\alpha(G_r)$ sep $\alpha(G)$.*

LEMMA 3. *Let P, Q and R be polynomials, such that P sep R and Q sep R and let P and Q have equal degrees m. Then for A_1 and A_2 being arbitrary positive constants.*

$$\min\{p_i, q_i\} \le s_i \le \max\{p_i, q_i\}, \tag{5}$$

where p_i, q_i *and* s_i *are the zeros of* P, Q *and* $S = A_1P + A_2Q$*, respectively,* $i = 1, 2, \dots, m$.

Proof. Let T be the greatest common divisor of P and Q and let $P = T \cdot P_0$ and $Q = T \cdot Q_0$. Then also $S = T \cdot S_0$ with $S_0 = A_1P_0 + A_2Q_0$.

If $p_i = q_i$, then the inequalities (5) hold in a trivial manner. It is, therefore, sufficient to prove (5) for the zeros of P_0, Q_0 and S_0,

Let $p_{0,i}, q_{0,i}$ and $s_{0,i}$, $i = 1, \dots, m_0$ be the zeros of P_0, Q_0 and S_0, respectively, labelled in non-decreasing order.

Two cases are to be distinguished: m_0, the degree of P_0, Q_0 and S_0, is either even or odd. Here we shall suppose that m_0 is even; the proof for the case when m_0 is odd is fully analogous.

If m_0 is even, then for $x < p_{0,1}$ (respectively for $x < q_{0,1}$), the polynomial P_0 (respectively Q_0) has positive values. Therefore S_0 is necessarily positive for $x < \min\{p_{0,1}, q_{0,1}\}$. Similarly, in the interval $[\max\{p_{0,1}, q_{0,1}\}, \min\{p_{0,2}, q_{0,2})]$ the polynomial S_0 must be negative, in the interval $[\max\{p_{0,2}, q_{0,2}\}, \min\{p_{0,3}, q_{0,3}\}]S_0$ must be positive etc. Consequently, $s_{0,i}$ lies in the internal $[\min\{p_{0,i}, q_{0,i}\}, \max\{p_{0,i}, q_{0,i}\}]$, $i = 1, \dots, m_0$. The requirements P sep R and Q sep R quarantee that the above intervals will not overlap. $\square$

3. Some separation relations. From Lemma 2 we see that the polynomials $\varphi(G_r)$ and $\alpha(G_r)$ meet the requirements of Lemma 3. Hence we have the following immediate consequence of Lemma 3.

LEMMA 4. *For all* A, B *and* G,

$$\min_{r\in B}\{x_i^r\} \le x_i^* \le \max_{r\in B}\{x_i^r\} \qquad \min_{r\in B} \le y_i^* \le \max_{r\in B}\{y_i^r\}, \quad i = 1, \dots, n-1.$$

A special case of the result above is obtained by taking into account eqs. (3) and (4).

COROLLARY. *If* x_i' *and* y_i' *are the zeros of the first derivative of* $\varphi(G)$ *and* $\alpha(G)$*, respectively, then for* $i = 1, \dots, n-1$,

$$\min_r\{x_i^r\} \le x_i' \le \max\{x_i^r\}, \qquad \min_r\{y_i^r\} \le y_i' \le \max_r\{y_i^r\}.$$

THEOREM 2. *For all* A, B *and* G; $\varphi(G)$ *sep* $\varphi(G)$ *and* $\alpha^*(G)$ *sep* $\alpha(G)$

Proof. By eqn. (1) (or by Lemma 2), $x_i \le \min\{x_i^r\}$ and $\max\{x_i^r\} \le x_{i+1}$. Then by Lemma 4, $x_i \le x_i^* \le x_{i+1}$, and the first part of Theorem 2 follows. The proof of the second part is analogous. $\square$

THEOREM 3. *For all* A, B *and* G,

$$\varphi^*(G) \text{ sep } \varphi^-(G), \;\; \varphi^*(G) \text{ sep } \varphi^+(G), \;\; \alpha^*(G) \text{ sep } \alpha^-(G), \;\; \alpha^*(G) \text{ sep } \alpha^+(G).$$

Proof. We prove here only the first of the four statements given in the theorem, assuming besides that n is even. The proof in the case of odd n, as well as the proof of the additional three separation relations, follows in a completely analogous manner.

Let us further assume that the zeros of $\varphi(G)$ and $\varphi^*(G)$ are mutually distinct. (When this is not the case, then we have to find the greatest common divisor of $\varphi(G)$ and $\varphi^*(G)$ and to proceed similarly as in the proof of Lemma 3.) We already know that $\varphi^*(G)$ sep $\varphi(G)$. Since n is assumed to be even, for $x < x_1$ the polynomial $\varphi(G)$ has positive values, where as for $x < x_1^*$, $\varphi^*(G)$ is negative. Furthermore, $x_1 < x_i^*$. Therefore, $\varphi(G) - \varphi^*(G)$ will be positive for $x < x_1$. Similar arguments show that $\varphi(G) - \varphi^*(G)$ will be negative in the interval $[x_1^*, x_2]$, positive in the interval $[x_2^*, x_3]$ etc. Therefore, the zeros of $\varphi^-(G)$ lie in the intervals $[x_i, x_i^*]$, $i = 1, \dots, n-1$ and another zero lies in $[x_n, \infty)$. These intervals cannot overlap because of Theorem 2.

This proves that $\varphi^*(G)$ separates $\varphi(G)$ if n is even. □

THEOREM 4. *For all A, B and G,*

$$\varphi^-(G) \text{ sep } \varphi(G), \quad \varphi(G) \text{ sep } \varphi^+(G), \quad \alpha^-(G) \text{ sep } \alpha(G), \quad \alpha(G) \text{ sep } \alpha^+(G).$$

Proof. In the proof of the previous theorem it was shown that the zeros x_i^- of $\varphi^-(G)$ lie in the interval $[x_i, x_i^*]$, i. e. $x_i \leq x_i^-$ for $i = 1, 2, \dots, n$ This, however, is just the first separation relation given in Theorem 4. Etc. □

THEOREM 5. *For all A, B and G,* $\varphi^-(G)$ sep $\varphi^+(G)$ *and* $\alpha^-(G)$ sep $\alpha^+(G)$.

Proof. Apply Lemma 1 to Theorem 3 and 4. Note that $\varphi^*(G)$ and $\alpha^*(G)$ play now the role of the polynomial S. □

By proving Theorems 2–5 we have, of course, also completed the proof of Theorem 1. It can be sean that Theorem 1 is, in fact, a consequence of the interlacing relations (1) and (2). It would be interesting to see if results similar to those given in Theorem 1 hold also for subgraphs obtained by deletion of more that one vertex from the graph.

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