PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 37 (51), 1985, pp. 23–24

AN APPLICATION OF NONSTANDARD ANALYSIS TO FUNCTIONAL EQUATIONS

Miodrag Rašković

Abstract. Using methods of nonstandard analysis it is proved that all measurable solutions of the equation $f(x+y) = g(f(x),\ f(y),\ x, y)$ (with g continuous) are continuous.

We suppose that the reader is acquainted with the basic facts of nonstandard analysis, including Loeb measure.

Let k be a positive integer, H an infinite integer, $T_k^H = \{-k,\ 1/H-k,\ 2/H-k, \dots, -2/H+k,\ -1/H+k, k\}$ and μ' the Loeb mesure obtained from counting measure on T_k^H. Let f_k map $[-k,k]$ into R and let F_k map T_k^H into ${}^*R$.

Definition 1. Function F_k is a lifting of the function f_k iff

$$\mu'(\{x : \mathrm{st}(F_k(x)) = f_k(\mathrm{st}(x))\}) = 0$$

Definition 2. Function F_k is a uniform lifting of the function f_k iff $\mathrm{st}(F_k(x)) = f_k(\mathrm{st}(x))$ for each $x \in T_K^H$.

The following theorems connect these notions with well known notions of continuity and measurability.

THEOREM 1. ([**1, 2**]) *Function f_k is Lebesgue measurable iff it has a lifting function F_k.*

THEOREM 2. [**2**] *Function f_k is continuous iff it has a uniform lifting unction F_k.*

We offer a new proof of the following theorem essentialy due to Hahn.

THEOREM 3. [**3**] *Let $f : R \to R$ be Lebesgue measurable, $g : R^4 \to R$ continuous and $f(x+y) = g(f(x),\ f(y),\ x, y)$. Then f is continuous.*

AMS Subject Classification (1980): Primary 03 H 05, 39 B 99.

Proof. Suppose $(T_h^{2k}, {}^*\mathcal{P}(T_H^{2k}), \mu)$ is an internal measure space with a counting measure μ and let μ'' be the corresponding Loeb measure.

Lebesque measurable function $f_{2k} = f \restriction [-2k, 2k]$ by Theorem 1 has a lifting function F_{2k}. Let $U = \{x \in T_H^{2k} : \mathrm{st}(F_{2k}(x)) = f_{2k}(\mathrm{st}(x))\}$. Obviously $\mu(T_h^2 k) = \mu'(U) = 4k$. Let also $A \in T_H^{2k}$ be internal set such that $\mu'(A) > 3k$. It is easy to show that for all $x \in T_k^H$, $(x - A) \cap A \neq \varnothing$. Hence we can define an improved lifting function.

$$F_k(x) = \min\{{}^*g(F_{2k}(y),\ F_{2k}(z), y, z) \mid x = y + z \wedge y,\ z \in A\}$$

for each $x \in T_k^H$. It follows immediately that F_k is the uniform lifting function for f_k, so by Theorem 2 function f_k is continuous. Therefore f is continuos too.

Finally let us remark that Theorem 3 can be generalized in several directions.

REFERENCES

[1] H. J. Keisler, *Hyperfinite probability theory and probability logic*, manuscript, Univ of Wisconsin, 1979.

[2] P. A. Loeb, *Converrsion from non-standard to standard measure spaces and applications in probability theory*, Trans. Amer. Math. Soc. **211** (1975), 113–122.

[3] E. Seneta, *Regularly Varing Functions*, Lecture Notes Math. 508, Springer-Verlag, Berlin-Heidelberg-New York, 1976.

Prirodno-matematički fakultet
34000 Kragujevac
Yugoslavia

(Received 19 04 1984)