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GRAPHS WITH THE REDUCED SPECTRUM IN THE UNIT INTERVAL

Aleksandar Torgašev

Abstract. By the reduced spectrum (r.s.) of a finite connected graph, we mean the set of all its eigenvalues with the maximal and the minimal eigenvalues excluded. In this paper we characterize all finite connected graphs having at least one positive and at least one negative eigenvalue in their reduced spectrum, whose r.s. lies in the unit interval $[-1,1)$.

1. Introduction. Let G be any finite connected graph with n vertices. Its spectrum $\sigma(G) = \{\lambda_1, \lambda_2, \dots, \lambda_n\}$ $(\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n)$ is the set of all eigenvalues of its $0-1$ adjacency matrix.

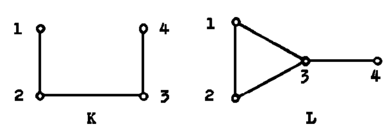
Its reduced spectrum $\sigma_0(G)$ is conditionally defined by

$$\sigma_0(G) = \{\lambda_2, \dots, \lambda_{n-1}\}.$$

Throughout the paper we assume that $n \geq 4$, $\lambda_2 > 0$ and $\lambda_{n-1} < 0$, thus G has at least one positive and at least one negative eigenvalue in its r.s.

Obviously, $\lambda_2 > 0$ and $\lambda_{n-1} > 0$ hold if only if $\lambda_2 > 0$, thus iff G has at least two positive eigenvalues. So, by a result of Smith [2], we have

THEOREM 1. *G has the property $\lambda_2 > 0$ and $\lambda_{n-1} < 0$ if and only if it has one of the following graphs as an (induced) subgraph.*



Consequently, all graphs considered have one of the graphs K, L as a subgraph. Hence, two possible cases arise:

I. G has K (but not L) as a subgraph;

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II. G has L as a subgraph.

Hence, in this paper we characterize all finite connected graphs with the property I or II, whose r.s. lies in the unit interval $J = [-1, 1)$, i.e. whose spectrum, except eventually the maximal and the minimal eigenvalues, lies in this interval.

We notice that our original aim was to give a similar characterization for the *closed* unit interval $[-1, 1]$, but because of numerous complications, we concluded that this will be unusually hard.

Throughout the paper, we consequently apply the method of impossible subgraphs.

Call, first, any graph G with the property $\sigma_0(G) \subseteq J$ admissible (with respect to the mentioned property), and all other graphs – impossible. Then by the interlacing theorem [1, p. 19], we have that each (induced) subgraph of an admissible graph is admissible too. Hence, applying the technique of impossible subgraphs, we obtain informations about the structure of admissible graphs.

Throughout the paper, we need to know the spectra of all particular graphs with at most 8 vertices. For graphs with 4 or 5 vertices, we use the list of spectra from [1], for graphs on 6 vertices (112 graphs) an internal publication, and for graphs with 7 or 8 vertices the help of an IBM–360 computer of the Mathematical Institute in Beograd. Knowing these spectra, we speak about the particular admissible and impossible graphs with this number of vertices.

In the whole paper, by a white circle we denote an arbitrary subset of isolated vertices in G, and by a black circle – any complete subgraph of G. By a line joining such two circles, we denote the fact that all possible edges between these circles exist. The number over a circle will always denote the number of its vertices.

K_n, P_n, C_n, and $E_n (n \geq 1)$ will denote the complete graph, the path, the circuit on n vertices, and the graph consisting of n isolated vertices, respectively. For any $n \in N$, we put $\bar{n} = n + 1$.

Finally, if $a_1, \dots, a_m$ are arbitrary vertices of G, $(a_1, \dots, a_m)$ will denote the subgraph of G induced by these vertices.

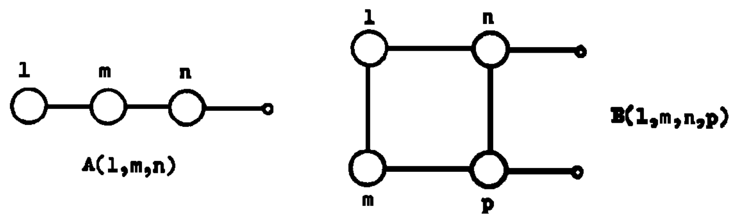
2. Case I. In this section we consider only the graphs having K (and not L) as an induced subgraph. Then we can apply a recent result from [3], where all connected graphs with the property $\lambda_{n-1} > -1$ are determined.

Lemma 1. *Each admissible G with the property I is bipartite.*

Proof. It is easily seen that in this case G has no C_3 as a subgraph. Bur since all $C_n (n \geq 5)$ are impossible graphs, G will not have any odd circuit as a subgraph, thus it will be bipartite. □

Hence, in this case we have $\lambda_{n-1} = -\lambda_2 > -1$, whence [3] can be applied. But since all graphs with the property $\lambda_{n-1} > -1$ are bipartite ([3]), whence $\lambda_2 > 1$, we get that G is admissible if and only if $\lambda_{n-1} > -1$. Consequently, we have

Theorem A. *Let $A(l, m, n)$, $B(l, m, n, p)$ $(l, m, n, p \in N)$ be the following two classes of graphs*



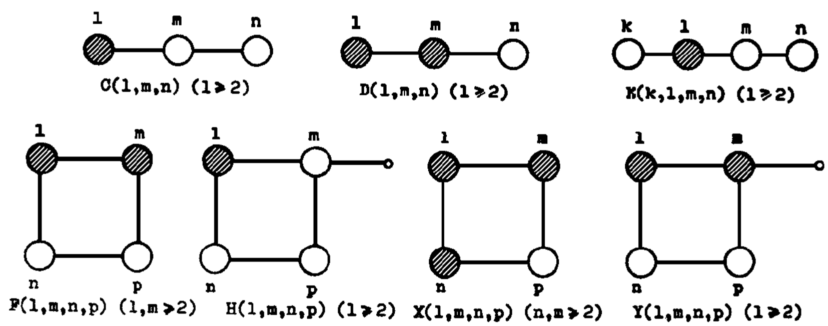
Then G, with the property I, is admissible if and only if

1° *$G = A(l,m,n)$, where $lmn + 1 < lm + mn + n$, or*

2° *$G = B(l,m,n,p)$, where $(n-1)(p-1) < (m+n-mn)(p+l-pl)$.* $\square$

3. Case II. In this section we describe all admissible graphs with L as an induced subgraph.

1. *Sufficiency.* Consider, first, the following 7 classes of graphs, and check when they give admissible graphs.



PROPOSITION 1. *The graph $C(l,m,n)$ $(l \geq 2)$ is admissible if and only if the relation $l(mn-m-1) \leq 2mn-3$ holds.*

Proof. As is easily seen, all the eigenvalues λ of $G = C(l,m,n)$ $(\lambda \neq 0; -1)$ are determined by the equation $(\lambda^2 - mn)\ (\lambda + 1 - l)) = ml\lambda$.

Whence the statement is immediate.

More precisely, it can be shown that G is admissible exactly in the following cases:

1° $l = 2,\ m,n \geq 1,$

2° $l = 3,\ n \leq 3,\ m \geq 1,$

3° $l = 4,\ n \leq 2,\ m \geq 1,$

4° $l = 5,\ m = n = 2,$

5° $l \geq 5,\ n = 1,\ m \geq 1,$

6° $l \geq 5,\ n = 2,\ m = 1.$ $\square$

PROPOSITION 2. *The graph $D(l,m,n)$ $(l \geq 2)$ is admissible if and only if the relation $(l-2)\ (mn+m-2) \leq ml-1$ holds.*

Proof. Since the corresponding equation for its eigenvalues $\lambda(\lambda \neq 0; -1)$ is $(\lambda+1-l)\ [\lambda(\lambda+1-m)-mn] = ml\lambda$, the statement is immediate.

The last relation is satisfied exactly in the following cases:

1° $l=2,\ m,n \geq 1;$

2° $l=3,\ n \leq 2, m \geq 1,$
$\quad n=3,\ m=1,$

3° $l=4,\ n=1,\ m \geq 1,$
$\quad n=2,\ m=1,$

4° $l=5,\ n=1,\ m=3,4,5,$

5° $l=6,7,\ n=1,\ m=3,$

6° $l \geq 4,\ n=2,\ m=1,$

7° $l \geq 5,\ n=1,\ m \leq 2.$ □

PROPOSITION 3. *The graph $E(k,l,m,n)$ $(l \geq 2)$ is admissible if and only if the relation $(mn-1)\ (kl+l-2) \leq ml-1$ holds.*

Proof. The corresponding equation will be

$$(\lambda^2 - mn)[\lambda(\lambda+1-l) - kl] = ml\lambda^2,$$

and all the rest is easy.

The above inequality is satisfied exactly in the following cases:

1° $n=k=1,\ l=2,\ m \geq 1,$
$\quad l=3,\ m \leq 3,$
$\quad l \geq 4,\ m \leq 2,$

2° $n=m=1,\ k \geq 1,\ l \geq 2.$ □

PROPOSITION 4. *The graph $F(l,m,n,p)$ $(l \geq m \geq 2)$ is admissible if and only if the relation $(l-2)\ (m-2) < [p(l-2)-l]\ [n(m-2)-m]$ holds.*

Proof. The corresponding equation will be

$$\lambda^2(\lambda+1-m)(\lambda+-l) = [(p+l)\lambda + p(1-l)][(m+n)\lambda + n(1-m)],$$

whence the statement follows.

The above relation is for $l=2,3,4$, satisfied exactly in the following cases:

1° $l=2,\ m=1,\ n,p \geq 1,$
$\quad m=2,\ n,p \geq 1,$

2° $l=3,\ m=2,\ p \leq 2,\ n \geq 1,$
$\quad m=3,\ p=1,\ n \leq 2,$
$\quad p=2,\ n=1,$
$\quad p=4,\ n \geq 5,$
$\quad p \geq 5,\ n \geq 4,$

3° $l=4,\ m=2,\ p=1,\ n \geq 1$
$\quad m=3,\ n=p=1$
$\quad p=3,\ n \geq 5$
$\quad p \geq 4,\ n \geq 4$
$\quad m=4,\ p=3,\ n \geq 4$
$\quad p \geq 4,\ n \geq 3.$ □

PROPOSITION 5. *The graph $H(l,m,n,p)$ $(l \geq 2)$ is admissible if and only if the relation $(m-1)\ (l-2) < (mn-n-m)\ (pl-l-2p)$ holds.*

Proof. The corresponding equation will be

$$\lambda^2(\lambda^2 - m)(\lambda + 1 - l) = [(m+n)\lambda^2 - mn][(p+l)\lambda + p(1-l)],$$

whence the statement. □

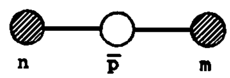
Concerning the class $X(l,m,n,p)$, we exclude the case $m = 1$, when X is an F-graph, moreover we suppose $n \geq m \geq 2$ $(l, p \geq 1)$. □

PROPOSITION 6. *The graph* $X(l,m,n,p)$ $(n \geq m \geq 2)$ *is always impossible.*

Proof. The corresponding equation will be

$$\lambda(\lambda+1-m)(\lambda+1-n)(\lambda+1-l) = [(p+l)\lambda + p(1-l)] \cdot [(m+n)\lambda + m + n - 2mn].$$

Hence, if $l = m = n = 2$, we get an impossible graph. Thus, we necessarily have $l = 1$. But then we obtain the graph



which has a similar induced subgraph with parameters $n = 2$, $\bar{p} = 2$, $m = 2$. Since the last subgraph is impossible, it follows that the considered graph is impossible too. □

Concerning the graph $Y(l,m,n,p)$ $(l \geq 2)$, we exclude the case $m = 1$ since $Y(l,1,n,p) = H(l,1,n,p)$.

PROPOSITION 7. *If* $m \geq 2$*, then the graph* $Y(l,m,n,p)$ $(l \geq 2)$ *is always impossible.*

Proof. The corresponding equation will be

$$\lambda^2(\lambda+1-l)(\lambda-m) = [(m+n)\lambda^2 + n\lambda(1-m) - mn] \cdot [(p+l)\lambda + p(1-l)].$$

Whence, if $l \geq 2$, $m \geq 2$, it is always an impossible graph. □

2. *Necessity.* Let G be any admissible graph with the property II. In the sequel, we shall prove that G is one of the graphs C, D, E, F, H, A special partition of the vertex set $V(G)$ will be an important step in this direction. Namely, let T^i $(i = 1,2,3,4)$ be the set of all vertices in G, outside of L, which are adjacent exactly to the vertex i. A similar meaning will be given to T_{ij}, T_{ijk} (i, j, k-distinct) and T_{1234}. The set of all these vertices is denoted by T (the first level of G). Similarly, we get the second level $\tilde{T}$ and the third level $\tilde{\tilde{T}}$ in G.

Later we shall see that $\tilde{\tilde{T}} = \emptyset$, so that each admissible graph has the property $V(G) = V(L) + T + \tilde{T}$ (+ denotes disjoint union).

Now, we want to determine the edge structure of al these parts as well as the edge structure between these parts. Therefore, for any such part A we choose a vertex $a \in A$ and test the subgraph $L + \{a\}$; then choose two vertices $a, b \in A$ and test the subgraph $L+\{a,b\}$ in the two possible cases (a, b adjacent or not). In this way we can conclude that, for example, A is empty, or consists only of isolated vertices, or is complete, or none of these things. Then we write $A/A=\emptyset$, 0, 1, $*$, respectively.

Similarly, for any two parts A, B, we write $A/B=\emptyset$, 0, 1, $*$. The last case is also called non-determined.

In this way, we easily obtain the following two propositions.

PROPOSITION 8. *All subsets in T, except eventually T_3, T_{12}, T_{123}, T_{124} and T_{1234} are empty.*

PROPOSITION 9. *The following relations hold: $T_3/T_3=0$, $|T_{12}\leq 1$, $T_{123}/T_{123}=1$, $T_{1234}/T_{1234}=1$, $T_{124}/T_{124}=*$.*

Therefore, in each admissible G, T_3 consists only of isolated vertices, and T_{123}, T_{124} are complete. Denote: $|T_3|=n$, $T_{123}=K_r$, $T_{1234}=K_m$ $(n,r,m\geq 0)$. The structure of T_{124} is determined in Proposition 10.

Concerning the edge structure between these subsets, we obtain the following table. Because of symmetry, this table is upper-triangular.

	T_3	T_{12}	T_{123}	T_{124}	T_{1234}
T_3		1	0	$*$	1
T_{12}			1	0	$\emptyset$
T_{123}				1	$\emptyset$
T_{124}					0
T_{1234}					

Next, we shall use the following abbreviations. Let P be any set of isolated vertices in G, and let $a\notin P$ be non adjacent to all $x\in P$, and with same neighbors as each $x\in P$. Then we write $a\sim P$, and can extend P to the wider white circle $P+\{a\}=P$.

Similarly, let Q be any black circle in G (i.e. a complete subgraph of G), and $a\notin Q$ be adjacent to each $x\in R$, and with the same neighbors as each $x\in Q$. Then we write $a\simeq Q$, and we can extend Q to the wider black circle $Q+\{a\}=\bar{Q}$.

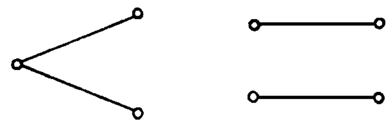
In the sequel, we investigate all the possible cases (excluding symmetric ones), arising when G consists of L and T only. We prove that in each of these cases G is one of the graphs C, D, E, F, H.

First, we discuss the structure of T_{124}. Let $M(p,q)=K_p+E_q$ $(p,q\geq 0)$ be the following disconnected graph:

Then we have

PROPOSITION 10. *In each admissible G, $T_{124} = M(p,q)$ holds.*

Proof. By testing all the possible subgraphs $H = L + T_{124}$ with three or four vertices in T_{124} we obtain that the following two subgraphs are impossible in T_{124}:



Hence, trivially, $T_{124} = M(p,q)$ $(p,q \geq 0)$.

Next, discussing this subgraph with general parameters p, q, we find that this graph is possible iff one of the following holds:

1° $p = 0$, $q \geq 1$;

2° $p = 2$, $q \geq 0$;

3° $p = 3$, $q = 0$. □

PROPOSITION 11. $L = C(2,1,1)$ *holds.*

PROPOSITION 12. *If $G = L + T_3$, then it is a C-graph.*

Proof. Since $4 \sim T_3$ in G, we obviously have that $G = C(2,1,\bar{n})$ $(\bar{n} \geq 2)$. □

PROPOSITION 13. *If $G = L + T_{12}$, then it is an E-graph.*

Proof. Obvious, since $G = E(1,2,1,1)$. □

PROPOSITION 14. *If $G = L + T_{123}$, then $G = C(r+2,1,1)$.*

PROPOSITION 15. *If $G = L + T_{124}$, then it is a C-graph or an F-graph.*

Proof. We have that $3 \sim E_q$, whence if $p = 0$ it follows $G = C(2,\bar{q},1)$ $(\bar{q} \geq 2)$, and $G = F(2,p,\bar{q},1)$ $(p,\bar{q} \geq 1)$ if $p \geq 2$. □

PROPOSITION 16. *If $G = L + T_{1234}$, then it is a C-graph.*

Proof. We have $3 \simeq T_{1234} = K_m$, and $1 \simeq 2$, whence $G = D(2,\bar{m},1)$ $(\bar{m} \geq 2)$. □

PROPOSITION 17. *If $G = L + T_3 + T_{12}$, then it is an H-graph.*

Proof. We obviously have that $G = H(2,1,1,n)$ where $n = |T_3| \geq 1$. □

PROPOSITION 18. *If $G = L + T_3 + T_{123}$, then it is a C-graph.*

Proof. Evidently, $4 \sim T_3$, $1,2 \simeq T_{123} = K_r$, hence $G = C(r+2,1,\bar{n})$ $(r \geq 1, \bar{n} \geq 2)$. □

PROPOSITION 19. *If $G = L + T_3 + T_{124}$, then one of the following cases occurs:*

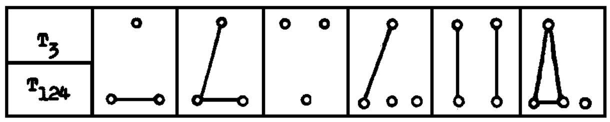
1° *$T_3/T_{124} = 0$, $n = 1$, $p = 0$, when G is a C-graph;*

2° *$T_3/T_{124} = 1$, when G is an F-graph;*

3° *$T_3/T_{124} = 1$, except exactly one edge, $p = 0$, when G is an H-graph;*

4° *$T_3/T_{124} = 1$ except all edges a_0b ($a_0 \in T_3$ fixed, and $b \in E_q \subseteq T_{124}$), $p = 0$, when G is an H-graph.*

Proof. First, consider the edge structure T_3/T_{124}. By testing all the possible subgraphs $L + T_3 + T_{124}$ with at most 3 vertices in T_3 and in T_{124}, we get the following six impossible subgraphs:



1 2 3 4 5 6

If $p \geq 2$, by the subgraphs (1), (2) and (4), we get that each $a \in T_3$ is adjacent to all $b \in K_p$ and to all $b \in E_q$, except eventually at most one $b_0 \in E_q$ (assuming that a is adjacent to at least one $b \in E_q$). In this case, denote by A the set of all $a \in T_3$ adjacent to all vertices $b \in T_{124}$, and by B the set of all $a \in T_3$ adjacent to all $b \in T_{124}$ except exactly one vertex of E_q. Then, by the impossible subgraphs (3) and (5), we conclude that $|B| \leq 1$. Therefore, if $|B| = 0$, we get that $T_3/T_{124} = 1$ except exactly one edge ab ($a \in T_3$, $b \in E_q$).

Next, let $p = 0$, and assume that T_3/T_{124} is not zero. Then each $b \in T_{124} = E_q$ is adjacent to all $a \in T_3$ except eventually an $a_0 \in T_3$. Let M be the set of all $b \in E_q$ adjacent to all vertices $a \in T_3$, and S the set of all $b \in E_q$ non adjacent to exactly one vertex from T_3. By the subgraph (5), we get that all $b \in S$ are non adjacent to exactly one vertex $a_0 \in T_3$.

If now $|S| = 1$, we get that $T_3/T_{124} = 1$ except exactly one edge ab ($a \in T_3$, $b \in E_q$). If otherwise, $|S| \geq 2$, by the subgraph (4), we conclude that $M = \emptyset$, thus $T_3/T_{124} = 1$, except all possible edges a_0b ($b \in E_q$).

In view of all these results, we conclude that only cases 1°–4° are possible. Next we discuss each of these cases separately.

1° If $T_3/T_{124} = 0$, then necessarily $p = 0$, and we obtain that $G = C(2, 1, q, 1)$ ($q \geq 1$).

2° If $T_3/T_{124} = 1$, then obviously $3 \sim E_q$, $4 \sim T_3$, whence we get that $G = F(2, p + 2, \bar{q}, \bar{n})$ ($p \geq 0$, $\bar{q} \geq 1$, $\bar{n} \geq 2$).

3° In this case, by the impossible subgraph (6), we conclude that $p = 0$, thus $T_{124} = E_q$ ($q \geq 1$). Then $3 \sim E_q \backslash \{b_0\}$, $4 \sim T_3 \backslash \{a_0\}$, whence $G = H(2, q, 1, n)$ ($q, n \geq 1$).

4° In this case $p = 0$ again, when $4 \sim T_3 \backslash \{a_0\}$, whence we get that $G = H(2, 1, q, n)$ ($q, n \geq 1$).

This completes the proof. □

PROPOSITION 20. *If* $G = L + T_3 + T_{1234}$, *then it is a* C*-graph.*

Proof. In this case obviously $3 \simeq T_{1234} = K_m$ and $4 \sim T_3$, whence $G = C(2, \bar{m}, \bar{n})$ $(\bar{m}, \bar{n} \geq 2)$. □

PROPOSITION 21. *If* $G = L + T_{12} + T_{123}$, *then* G *is an* E*-graph.*

Proof. Then $1, 2 \simeq T_{123} = K_r$, whence $G = E(1, r + 2, 1, 1)$. □

PROPOSITION 22. *If* $G = L + T_{12} + T_{124}$, *then* $p = 0$, *and* G *is an* E*-graph.*

Proof. Let $p \geq 2$, $a, b \in K_p$ and $c \in T_{12}$. Then the subgraph $(c12ab4)$ is impossible, contradiction. Hence necessarily $p = 0$, thus $T_{124} = E_q$ $(q \geq 1)$. But then $3 \sim E_q$, whence $G = E(1, 2, \bar{q}, 1)$ $(\bar{q} \geq 2)$. □

Proposition 23. If $G = L + T_{123} + T_{124}$, *then* G *is a* C*-graph or an* F*-graph.*

Proof. We have that $T_{124} = M(p, q)$ $(p, q \geq 0)$. Then $1, 2 \simeq T_{123} = K_r$, and $3 \sim E_q$, whence if $p = 0$, we get $G = C(r + 3, \bar{q}, 1)$, and if $p \geq 2$, we get $G = F(r + 3, p, \bar{q}, 1)$ $(p, \bar{q}, r \geq 1)$. □

PROPOSITION 24. *If* $G = L + T_{124} + T_{1234}$, *then* $p = 0$ *and* G *is an* F*-graph.*

Proof. In this case obviously $3 \simeq T_{1234} = K_m (m \geq 1)$. If $p \geq 2$, then choosing $a, b \in K_p$ and $c \in T_{1234}$ we get the impossible subgraph $(ab123c)$. Hence, necessarily, $p = 0$, whence $G = F(\bar{m}, 2, 1, q)$ $(\bar{m} \geq 2, q \geq 1)$. □

PROPOSITION 25. *If* $G = L + T_3 + T_{12} + T_{123}$, *then* G *is an* H*-graph.*

Proof. We have that $1, 2 \simeq T_{123} = K_r$, whence obviously $G = H(r + 2, 1, 1, n)$ $(r, n \geq 1)$. □

PROPOSITION 26. *If* $G = L + T_3 + T_{12} + T_{124}$, *then* $p = 0$, $T_3/T_{124} = 1$, *and* G *is an* F*-graph.*

Proof. By Proposition 22, we necessarily have $p = 0$. Next, let $a \in T_3$, $b \in E_q = T_{124}$ be a pair of non adjacent vertices, and let $c \in T_{12}$. Then the subgraph $(ac4b)$ is impossible, contradiction. Consequently, $T_3/T_{124} = 1$.

Now we obviously have that $4 \sim T_3$, $3 \sim E_q$, whence $G = F(2, 1, \bar{q}, \bar{n})$ $(\bar{q}, \bar{n} \geq 2)$. □

PROPOSITION 27. *If* $G = L + T_3 + T_{123} + T_{124}$, *then* G *is one of the graphs* C, F, H.

Proof. In view of the structure T_3/T_{124}, we distinguish four cases according to Proposition 19.

1° In this case $p = 0$, $n = 1$ whence $G = H(r + 2, 1, q, 1)$ $(r, q \geq 1)$.

2° Then $4 \sim T_3$, $3 \sim E_q$, $1, 2 \simeq T_{123}$, whence $G = C(r + 2, \bar{q}, \bar{n})$ $(\bar{q}, \bar{n} \geq 2)$ if $p = 0$, and $G = F(r + 2, p, \bar{q}, \bar{n})$ $(\bar{q}, \bar{n} \geq 2)$ if $p \geq 2$.

3° In this case $p=0$, $4\sim T_3\backslash\{a_0\}$, $3\sim T_{124}\backslash\{b_0\}$, whence $G=H(r+2,q,1,n)$ $(r,q,n\geq 1)$.

4° Then $p=0$, $4\sim T_3\backslash\{a_0\}$, whence we have that $G=H(r+2,1,q,n)$ $(r,q,n\geq 1)$. $\square$

PROPOSITION 28. *If $G=L+T_3+T_{124}+T_{1234}$, then $p=0$, and G is an F-graph.*

Proof. By Proposition 24 we necessarily have that $p=0$, whence $3\simeq T_{1234}=K_m$.

We are now proving that $T_3/T_{124}=1$. Let, in the opposite case, $a\in T_3$ and $b\in T_{124}$ be nonadjacent, and $c\in T_{1234}$. Since the subgraph $(1234abc)$ is impossible, we necessarily have that $T_3/T_{124}=1$. Then $4\sim T_3$, whence $G=F(2,\bar{m},q,\bar{n})$ $(\bar{m},\bar{n}\geq 2, q\geq 1)$. $\square$

PROPOSITION 29. *If $G=L+T_{12}+T_{123}+T_{124}$, then $p=0$, and G is an E-graph.*

Proof. By Proposition 22 we necessarily have $p=0$. But then $3\sim T_{124}=E_q$, whence $G=E(1,r+2,\bar{q},1)$ $(r\geq 1,\bar{q}\geq 2)$. $\square$

PROPOSITION 30. *If $G=L+T_3+T_{12}+T_{123}+T_{124}$, then $p=0$, $T_3/T_{124}=1$, and G is an F-graph.*

Proof. By Proposition 26 we obtain that $p=0$ and $T_3/T_{124}=1$. Then $4\sim T_3$, $3\sim E_q 1, 2\simeq T_{123}=K_r$, whence $G=F(\bar{q},\bar{n},r+2,1)$ $(\bar{q},\bar{n}\geq 2, r\geq 1)$. $\square$

In what follows, we investigate the structure of the second level $\tilde{T}$ in an arbitrary admissible G. By the method of impossible subgraphs, we immediately have

PROPOSITION 31. *In $\tilde{T}$ only remains $\tilde{T}_{124}$.*

PROPOSITION 32. $\tilde{\tilde{T}}_{124}=\emptyset$.

Whence, the third level $\tilde{\tilde{T}}$ in each admissible G is empty.

LEMMA 2. *Each $x\in\tilde{T}_{124}$ is non adjacent to every $a\in K_p\subseteq T_{124}$.*

Proof. Let x be adjacent to some $a\in K_p\subseteq T_{124}$, and let $b\in K_p$. Then if x, b are adjacent, we get the impossible subgraph $(321abx)$, while otherwise, we obtain the impossible subgraph $(12ab4x)$.

Hence, each $x\in\tilde{T}_{124}$ is adjacent only to some vertices of $E_q\subseteq T_{124}$. $\square$

LEMMA 3. *Two vertices $x,y\in\tilde{T}_{124}$ cannot be adjacent to the same vertex $a\in T_{124}$.*

Proof. Let, in the opposite case, $x,y\in\tilde{T}_{124}$ be adjacent to the same vertex $a\in T_{124}$. Then, if x, y are adjacent, we get the impossible subgraph $(234axy)$, while otherwise we obtain the impossible subgraph $(312axy)$. $\square$

Proposition 33. $|\tilde{T}_{124}| \leq 1$.

Proof. Let, in the opposite case, $x, y \in \tilde{T}_{124}$ be adjacent to the vertices $a, b \in T_{124}$, respectively. Then x, b as well as y, a are non adjacent. Therefore, if x, y are adjacent, we obtain the impossible subgraph $(xa4by) = C_5$, while otherwise we get the impossible subgraph $(xa4by) = P_5$.

Proposition 34. *If $G = L + T_{124} + \tilde{T}_{124}$, then $p = 0$, and G is an H-graph.*

Proof. If, in the opposite case, $p \geq 2$, choose $x \in \tilde{T}_{124}$ adjacent to a vertex $a \in E_q \subseteq T_{124}$, then $b, c \in K_p \subseteq T_{124}$, when we obtain the impossible subgraph $(bc12ax)$. Therefore $p = 0$.

Next, put $E_1 = \{a \in E_q \mid x, a \text{ adjacent}\}$, $E_0 = E_q \backslash E_1$ and $q_1 = |E_1| \geq 1$, $q_0 = |E_0| \geq 0$. Then $3 \sim E_0$, whence $G = H(2, q_1, q_0, 1)$ $(q_1, q_0 \geq 1)$. □

Proposition 35. *Let $G = L + T_3 + T_{124} + \tilde{T}_{124}$. Then $p = 0$, $T_3/T_{124} = 1$, and G is an H-graph.*

Proof. By the previous proposition, we have that $p = 0$. Assuming next that $a \in E_1$ and $b \in T_3$ are not adjacent, we obtain the impossible subgraph $(b34ax) = P_5$. Similarly, assuming that $b \in T_3$ and $c \in E_0$ are non adjacent, and then choosing an $a \in E_1$ and $x \in \tilde{T}_{124}$, we conclude that a, b are adjacent, whence the impossible subgraph $(c12abx)$ arises. Consequently $T_3/E_1 = T_3/E_0 = 1$ thus $T_3/T_{124} = 1$. Hence $4 \sim T_3$, $3 \sim E_0$, whence denoting $q_1 = |E_1| \geq 1$, $q_0 = |E_0| \geq 0$, we obtain that $G = H(2, q_1, \bar{q}_0, \bar{n})$ $(q_1, \bar{q}_0 \geq 1, \bar{n} \geq 2)$. □

Proposition 36. *$G = L + T_{12} + T_{124} + \tilde{T}_{124}$ is always impossible.*

Proof. Choosing $a \in T_{12}$, $b \in E_1 \subseteq T_{124}$ and $x \in \tilde{T}_{124}$, we obtain the impossible subgraph $(a12bx)$. □

Proposition 37. *If $G = L + T_{123} + T_{124} + \tilde{T}_{124}$, then G is an H-graph.*

Proof. By Proposition 34 we get $p = 0$. But then $1, 2 \simeq T_{123} = K_r$, $3 \sim E_0$, whence $G = H(r + 2, q_1, \bar{q}_0, 1)$ $(r, q_1, \bar{q}_0 \geq 1)$. □

Proposition 38. *$G = L + T_{1234} + T_{124} + \tilde{T}_{124}$ is always impossible.*

Proof. Choosing $x \in \tilde{T}_{124}$, $a \in E_1 \subseteq T_{124}$, $b \in T_{1234}$, we get the impossible subgraph $(123xab)$. □

Therefore, it remains only to test the graph $L + T_3 + T_{123} + T_{124} + \tilde{T}_{124}$,

Proposition 39. *If $G = L + T_3 + T_{123} + T_{124} + \tilde{T}_{124}$, then G is an H-graph.*

Proof. By Proposition 35, we have that $p = 0$, $T_3/T_{124} = 1$. But then $3 \sim E_0$, $4 \sim T_3$, whence $G = H(r + 2, q_1, \bar{q}_0, \bar{n})$ $(r, q_1, \bar{q}_0 \geq 1, \bar{n} \geq 2)$, which completes the proof. □

Summarizing all the propositions 11–39, we conclude with the following main theorem:

THEOREM B. *In case II, each admissible graph G is one of the graphs C, D, E, F, H.*

In such a way, by Theorems A and B, having in mind Proposition 1 – 5, we characterized all connected graphs whose reduced spectrum lies in the interval $[-1, 1)$.

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Institut za matematiku
Prirodno-matematički fakultet
Studentski trg 16a
11000 Beograd, Jugoslavija

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