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ON RANDOM VARIABLES WITH THE SAME DISTRIBUTION TYPE AS THEIR RANDOM SUM

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Abstract. Let $\xi_1, \xi_2, \dots, \xi_n, \dots$ be a sequence of nonnegative, independent, equally distributed random variables with distribution function $F(x)$ and corresponding Laplace transform $f(t)$; let ν be integer-valued random variable independent of ξ_n, $n = 1, 2, \dots$, $p_n = P(\nu = n)$, $p_0 = 0$, $P(s) = \sum_{n=0}^{\infty} s^n p_n$ – its generating function. In this paper, solutions (P, f) of the following functional equation are found:

$$P(f(t)) = f(c_\nu t),$$

where c_ν is a real number depending on ν.

Let $\xi_1, \xi_2, \dots, \xi_n, \dots$ be a sequence of nonnegative, independent and equally distributed random variables with distribution function $F(x) = P\ (\xi_1 < x)$, and ν integer-valued random variable, $p_n = P(\nu = n)$, $p_0 = 0$, independent of the random variables ξ_k, $k = 1, 2, \dots$. Denote by S_ν the following sum:

$$S_\nu = \xi_1 + \xi_2 + \cdots + \xi_\nu,$$

$\Phi_\nu(x) = P(S_\nu x)$. Obviously, we have

$$\Phi_\nu(x) = \sum_{n=1}^{\infty} P(S_n < x) p_n = \sum_{n=1}^{\infty} F^{*(n)}(x) p_n \tag{1}$$

where $F^{*(n)}(x)$ is the distribution function of the sum $S_n = \xi_1 + \xi_2 + \cdots + \xi_n$. In terms of Laplace transform (1) becomes

$$\varphi(t) = \sum_{n=1}^{\infty} (f(t))^n p_n, \tag{2}$$

where $f(t) = \int\limits_0^\infty e^{-tx} dF(x)$, $\varphi(t) = \int\limits_0^\infty e^{-tx} d\Phi_\nu(x)$.

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Let us denote by K the class of random variables for which

$$\Phi_\nu(x) = F(x/c_\nu) \tag{3}$$

is valid, where c_ν is a real number depending on ν. In other words, K is the class of random variables for which the distribution function of the sum $\xi_1 + \cdots + \xi_n u$ of random number of equally distributed random variables differs only by a scale factor from the distribution of one random variable from that sum, i.e., the distribution function of random variables from K has the same type [**1**, p. 137] as the distribution of one random variable from the sum.

From (2) and (3) we have the functional equation

$$\sum_{n=1}^{\infty} (f(t))^n p_n = f(c_\nu t) \tag{4}$$

Let us denote by $P(s)$ the generating function of random variable ν, $P(s) = \sum_{n=0}^{\infty} s^n p_n$, $s \in (0,1]$; then (4) becomes

$$P(f(t)) = f(c_\nu t). \tag{5}$$

In [**2**] a case when the random variable ν has geometric distribution (i.e. $p_n = pq^{n-1}$, $p+q=1$, $n=1,2,\dots$) was investigated; then equation (5) reduces to

$$pf(t)(1-qf(t))^{-1} = f(c_p t),$$

and a characterization of the corresponding class of random variables was given by their Laplace transforms.

The question is how the solutions (P,f) of functional equation (5) look like, where P and f are, as before, generating function and Laplace transform of some random variables. We shall obtain the solution of (5) as a special case when $s=1$ from the solution of the more general functional equation:

$$P(sf(t))/P(s) = f(c(s)t), \tag{6}$$

$t \in [0,\infty)$, $s \in (0,1]$, where $P(s)$ is the generating function of a nondegenerate, integer-valued random variable with finite mathematical expectation and variance, $f(t)$ is the Laplace transform of nondegenerate random variable; $c(s)$ is a real function on $(0,1]$.

THEOREM. *Functions* (P,f,c) *are solutions of the functional equation* (6) *if and only if*

$$\begin{aligned} P(s) &= \left[\frac{ps^k}{1-(1-p)s^k}\right]^{1/k}, \quad p \in (0,1), \ \ k \in N, \ \ s \in (0,1], \\ f(t) &= (1+ct^\alpha)^{-1/k}, \ \ c>0, \ \ 0<\alpha\le 1, \ \ t \ge 0, \\ c(s) &= (1-(1-p)s^k)^{-1/\alpha}. \end{aligned} \tag{7}$$

Proof. In one direction this theorem is proved immediately by substitution of (7) in the equality (6).

We shall now prove the converse of the theorem. Let us suppose that functional equation (6) is valid for some functions $P(s)$, $f(t)$ and $c(s)$. Since $f(t)$ is a Laplace transform, it must be strictly monotone, continuous and it possesses derivatives of all orders. Let us show that its first derivative never equals zero. If, on the contrary, $f'(t_0) = 0$ for some t_0, then for every $T > 0$ the following holds:

$$0 = f'(t_0) = \int_0^\infty xe^{-t_0 x} dF(x) \geq \int_T^\infty xe^{-t_0 x} dF(x) \geq$$
$$\geq T(1 - F(T))e^{-t_0 x_0}, \;\; x_0 \in (T, \infty),$$

so we have that $F(x)$ is a distribution function of a degenerate random variable, contrary to our assumption. Since the first derivative of $f(t)$ never vanishes, the derivative of the inverse function $f^{-1}(u)$ exists.

Let us write (6) in the following way:

$$f^{-1}(P(su)/P(s)) = c(s) f^{-1}(u), \quad u \in (0,1), \quad s \in (0,1]. \tag{8}$$

If, for some s_0, $c(s_0) = 0$, it follows from (6) that $f(t) \equiv 1$, which is impossible. So, $c(s) \equiv 0$, for every $s \in (0,1]$.

From (8) we have

$$\frac{f^{-1}(P(su)/P(s))}{f^{-1}(P(sv)/P(s))} = \frac{f^{-1}(u)}{f^{-1}(v)}, \quad u, v \in (0,1), \;\; s \in (0,1]. \tag{9}$$

Differentiating (9) with respect to s, u, v, after simple transformations we get

$$\frac{f'(f^{-1}(u) f^{-1}(v)}{f'(f^{-1}(v) f^{-1}(u)} = \frac{P'(sv)(uP'(su)P(s) - P'(s)P(su))}{P'(su)(vP'(sv)P(s) - P'(s)P(sv))}. \tag{10}$$

Let us differentiate (10) with respect to s, and let s tend to 1. We have

$$\frac{uP''(u)}{P'(u)} - \frac{u^2 P''(u) - P''(1)P(u)}{uP'(u) - P'(1)P(u)} = \frac{vP''(v)}{P'(v)} - \frac{v^2 P''(v) - P''(1)P(v)}{vP'(v) - P'(1)P(v)} \tag{11}$$

for every $u, v \in (0,1)$. If, for fixed v, we denote by a the expression on the right-hand side of equality (11), we have

$$P'(1)P''(u)/P'(u) + aP'(u)/P(u) - (P''(1) + aP'(1))/u = 0, \;\; u \in (0,1), \;\; P(1) = 1$$

which is equivalent to

$$P'(1) \ln P'(u) + a \ln P(u) - (P''(1) + aP'(1)) \ln u = C_0 = P'(1) \ln P'(1).$$

The solution of this equation is:

$$P(u) = \left[\frac{(a/P'(1) + 1)P'(1)}{a + P''(1)/P'(1) + 1} u^{a + P''(1)/P'(1) + 1} + C_1 \right]^{a/P'(1) + 1)^{-1}}.$$

From $P(1) = 1$ we obtain that

$$C_1 = 1 - \frac{(a/P'(1) + 1)P'(1)}{a + P''(1)/P'(1) + 1}.$$

Solution $P(u)$ is of the form

$$P(u) = [p^{-1}u^{-\alpha} - (1-p)p^{-1}]^{-\beta/\alpha}, \quad u \in (0,1). \tag{12}$$

Let us show that α, β are positive integers and $0 < p < 1$.
1. If $\alpha < 0$, let us write the Taylor expansion for $P(u)$:

$$P(u) = \left(\frac{p-1}{p}\right) p^{-\beta/\alpha} \left(1 + \frac{u^{-\alpha}}{p-1}\right)^{-\beta/\alpha} = \left(\frac{p-1}{p}\right)^{-\beta/\alpha} \sum_{n=0}^{\infty} \binom{-\beta/\alpha}{n} \left(\frac{u^{-\alpha}}{p-1}\right)^{n}.$$

Since $P(u)$ is a generating function, powers of u must be positive integers, which means that $-\alpha \in N$; for all n, coefficients must be nonnegative, i.e., $\binom{-\beta/\alpha}{n}(p-1)^{-n} \geq 0$, $n = 0, 1, \dots$, whence $\beta < 0$ and $p < 1$. But then we get $P(u) > 1$, which is impossible.
2. If $\alpha > 0$, the Taylor expansion for (12) is

$$\begin{aligned} P(u) &= (u^{-\alpha}p^{-1})^{-\beta/\alpha}(1 + (p-1)u^{\alpha})^{-\beta/\alpha} = \\ &= (u^{-\alpha}p^{-1})^{-\beta/\alpha} \sum_{n=0}^{\infty} \binom{-\beta/\alpha}{n} (p-1)^n u^{\alpha n} = \sum_{n=0}^{\infty} p^{\beta/\alpha} \binom{-\beta/\alpha}{n} (p-1)^n u^{\alpha n + \beta}, \end{aligned}$$

whence again $\beta > 0$, $p < 1$, α and β are positive integers. From $p < 0$ it follows that $P(u) > 1$, and we must have $p > 0$. We obtain that $P(u)$ is

$$P(u) = \left[\frac{pu^k}{1 - (1-p)u^k}\right]^{n_1/k}, \quad u \in (0,1), \ p \in (0,1), \ k, n_1 \in N.$$

Let us write $M(t) = (f(t))^k$, $(1-p)s^k = 1 - z$; then from (6):

$$\left[\frac{zM(t)}{1 - (1-z)M(t)}\right]^{n_1} = M(\zeta(z)t), \quad t \in R^+, \tag{13}$$

where $\zeta(z)$ is a real function on $(p, 1)$,

$$\zeta(z) = t^{-1}M^{-1}\left(\left[\frac{zM(t)}{1 - (1-z)M(t)}\right]^{n_1}\right),$$

and therefore $\zeta(z)$ is differentiable and its derivative never vanishes. Let us differentiate (13) with respect to t and z; after simple transformations we get

$$\frac{tM'(t)}{M(t)(1 - M(t))} = \frac{\zeta(z)}{z\zeta'(z)}, \quad t \in (0, \infty), \ z \in (p, 1). \tag{14}$$

If we denote by a_0 the left-hand side of (14) for t fixed, it follows that $a_0/z = \zeta'(z)/\zeta(z)$, whence $\zeta(z) = z^{a_0}/a_1$. In order to obtain $M(t)$, let us write (14) in the following way (for fixed z, the right-hand side of (14) is constant b_0):

$$\frac{b_0}{t} = \frac{M'(t) - 2M(t)M'(t) + 2M(t)M'(t)}{M(t)(1 - M(t))}.$$

It follows that

$$b_0 \ln t = \ln M(t)(1 - M(t)) - 2\ln(1 - M(t)) + \ln C,$$

and therefore $M(t) = (1 + ct^{-b_0})^{-1}$. Since $M(t)$ is an k-th power of the Laplace transform $f(t)$, we must have $c > 0$, $-b_0 = \alpha \in (0, 1]$. So, $M(t)$ and $\zeta(z)$ are solutions of equation (13) only when $n_1 = 1$, $a_1 = 1$, $a_0 = -1/\alpha$. We have that

$$f(t) = (1 + ct^{\alpha})^{-1/k}, \;\; c(s) = [1 - (1 - p)s^k]^{-1/\alpha}.$$

The proof is complete.

We supposed that $P(s)$ was the generating function of a nondegenerate random variable. Let us discuss now solutions of (6) in case $P(s)$ is the generating function of a degenerate random variable. Then (6) reduces to

$$(f(t))^n = F(ct), \;\; t \in [0, \infty), \;\; n \in N, \;\; c > 1. \tag{15}$$

If $c = 1$, then if $n = 1$, f is arbitrary, and if $n > 1$, f is the Laplace transform of a degenerate random variable. Let us consider the case when $n > 1$, $c > 1$. Since for every $t \in [0, \infty)$ $f(t) \neq 0$, we can write $f(t) = \exp(u(t))$, and (15) becomes $\exp(nu(t)) = \exp(u(ct))$, or

$$nu(t) = u(ct). \tag{16}$$

Let $u_1(t)$ and $u_2(t)$ be two solutions of (16), and let $g(t) = u_1(t)/u_2(t)$. Then, from (16), we have

$$g(t) = u_1(t)/u_2(t) = u_1(ct)/u_2(ct) = g(ct).$$

Let $h(\ln t) = g(t)$, then $h(\ln t) = h(\ln t + \ln c)$, i.e. $h(s) = h(s + \ln c)$. Whence $u(t) = u_0(t)h(\ln t)$ and $f(t) = (f_0(t))hc \ln t) = \exp(u_0(t)h(\ln t))$. Since (15) holds,

$$(f(t))^n = (f_0(t))^{nh(\ln t)} = (f_0(ct))^{h(\ln t)} = (f_0(ct))^{h(\ln t + \ln c)} = f(ct).$$

One solution of (15) is $f_0(t) = \exp(-t^{\beta})$, $0 < \beta < 1$. Let us compute β. We have $u_0(t) = -t^{\beta}$, $-nt^{\beta} = -(n^{1/\beta}t)^{\beta} = u_0(ct)$; then $c = n^{1/\beta}$, whence $\beta = \ln n / \ln c$. Since $f(t) = \exp(-t^s h(\ln t))$, then $h(\ln t) = -t^s \ln f(t)$. When $h(\ln t)$ is constant, $f(t)$ is the Laplace transform of a stable, [**1**, p. 448] distribution function.

Remark. It is natural to ask about solutions of (6) when $P(s)$ and $c(s)$ are as before, and $f(t)$ is the characteristic function of a random variable which need not be only positive. In [**3**], the solution of (4) is found on the condition that ν has the geometric distribution function and $f(t)$ is the characheristic function of a random variable with arbitrary sign. It is easy to see that if $P(s)$ and $c(s)$ are as in (7) and

$$f(t) = (1 + 4v(t))^{-1/k}$$

where $v(t) = (c_0 + it/|t|c_1|)|t|^\alpha$, $0 < \alpha \leq 2$, $c_0 \geq 0$, $c_1 \in R$, $k \in N$, then stch (P, f, c) are solutions of the equation (6). The question is whether, except for these (P, f, c), other solutions in this class of function exists.

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