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SOME SPECIAL PRODUCT SEMISYMMETRIC AND SOME SPECIAL HOLOMORPHICALLY SEMISYMMETRIC F-CONNECTIONS

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Abstract. In the present paper we investigate two special product semisymmetric F-connections: PS-concircular and PS-coharmonic ones and we fined the conditions for product semisymmetric connection to be PS-concircular or PS-coharmonic. In the same manner we investigate two special holomornhically semisymmetric F-connections.

Introduction. The connection of an n-dimensional differentiable manifold is termed semisymmetric if its torsion tensor S satisfies

$$S^i_{jk} = \delta^i_j S_k - \delta^j_k S_j. \tag{0.1}$$

A semisymmetric connection is generalized on a locally decomposable Riemannian space in [**4**] and [**5**]. In [**4**] the product semisymmetric metric F-connection is defined and studied. In [**5**] something similar is done for the holomorphically semisymmetric F-connection.

In the present paper we investigate two special holomorphically semisymmetric and two special product semisymmetric F-connections. In **1** we recall what is a product semisymmetric F-connection. In **2** we define PS-concircular connections and prove that each of the relations (2.7), (2.9) and (2.17) is a necessary and sufficient condition for a product semisymmetric connection to be PS-concircular. In **3** we define PS-coharmonic connections and prove that a product semisymmetric connection is PS-coharmonic iff (3.3) holds. In **4** we recall what is a holomorphically semisymmetric connection. In **5** we define HS-concircular connection and prove that each of the. relations (5.6) and (5.8) is a necessary and sufficient condition for a holomorphically semisymmetric connection to be HS-concircular. In **6** we investigate another HS-connection.

These results generalize, for the locally decomposable Riemannian space, the results obtained by P. Strave in [**6**].

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1. Product semisymmetric metric F-connection. An n-dimensional differenciable manifold M_n of class C^∞ is called a locally decomposable Riemannian space [**8**] if in M^n a tensor field $F^i_j \neq \delta^i_j$ and a positive definite Riemannian metric $ds^2 = g_{ij}(x^k)dx^i dx^j$ are given, satisfying the conditions

$$F^i_k F^k_j = \delta^i_j, \quad g_{ab} F^a_i F^a_j = g_{ij}, \quad \nabla_k F^i_j = 0, \tag{1.1}$$

where ∇_k is the operator of the covariant derivative with respect to the Riemannian metric. If we put $F^a_i g_{aj} = F_{ij}$, then $F_{ij} = F_{ji}$ and the condition $\nabla_k F^i_j = 0$ is equivalent to the condition $\nabla_k F_{ij} = 0$.

Locally decomposable space can be covered by a separating coordinate system, that is, by such a system of coordinate neighborhoods (x^i) that in any intersection of two coordinate neighborhoods (x^i) and $(x^{i'})$ we get

$$x^{a'} = x^{a'}(x^a), \qquad x^{y'} = x^{y'}(x^y),$$

where the indices a, b, c run over the range $1, 2, \ldots, p$ and the indices x, y, z run over the range $p+1, \ldots, p+q = n$.

With respect to a separating coordinate system, the metric of the space has the form $ds^2 = g_{ab}(x^c)dx^a dx^b + g_{xy}(x^z)dx^x dx^y$, while the tensors F_{ij} and F^i_j have the forms:

$$(F_{ij}) = \begin{pmatrix} g_{ab} & 0 \\ 0 & -g_{xy} \end{pmatrix}, \quad \begin{pmatrix} \delta^b_a & 0 \\ 0 & -\delta^x_y \end{pmatrix}.$$

Therefore

$$\varphi \equiv p - q. \tag{1.2}$$

In the following we suppose $p > 2$, $q > 2$.

The product semisymmetric metric F-connection (PS-connection) has the form [**4**]:

$$\Gamma^i_{jk} = \left\{ {i \atop jk} \right\} + \delta^i_j S_k - g_{jk} S^i + F^i_j S_a F^a_k - S^a F^i_a F_{jk}, \tag{1.3}$$

where S_i is a decomposable vector field in M_n, i.e. the field satisfying the condition

$$F^a_j \nabla_a S + i = F^a_i \nabla_j S_a, \tag{1.4}$$

and $S^i = g^{ia} S_a$. This condition can be expressed in the form

$$F^a_i F^b_j \nabla_b S_a = \nabla_j S_i. \tag{1.5}$$

Also, we suppose in the following that S_i is locally a gradient vector field.

With respect to a separating coordinate system, all the Γ^i_{jk} are zero except

$$\Gamma^a_{bc} = \left\{ {a \atop bc} \right\} + 2S_c \delta^a_b - 2g_{bc} S^a, \quad \Gamma^x_{yz} = \left\{ {x \atop yz} \right\} + 2S_z \delta^x_y - 2g_{zy} S^x,$$

that is, both connections Γ^a_{bc} and Γ^x_{yz} are semisymmetric metric connections: the first with respect to the metric $g_{ab}(x^c)dx^a dx^b$, the second with respect to the metric $g_{xy}(x^z)dx^x dx^y$.

The curvature tensor R^i_{rkj} of the connection (1.3) can be expressed in the form [**4**]

$$\begin{aligned}R^i_{rkj} = K^i_{rkj} + \delta^i_j\rho_{rk} - g_{jr}\rho^i_k + F^i_j F^a_r \rho_{ak} - F_{jr} g^{ib} F^a_b \rho_{ak} - \\ - \delta^i_k \rho_{rj} + g_{kr}\rho^i_j - F^i_k F^a_r \rho_{aj} + F_{kr} g^{ib} F^a_b \rho_{aj},\end{aligned} \tag{1.6}$$

where

$$\rho_{rk} = \nabla_r S_k - S_r S_k + S^a S_a g_{rk}/2 - S_a S_b F^a_r F^b_k + S^a S_b F^b_a F_{rk}/2, \tag{1.7}$$

and K^i_{rkj} is the curvature tensor of the Riemannian space having g_{ij} as a metric tensor. Since S_i is a gradient, tensor ρ_{rk} is symmetric. Also, it satisfies the condition

$$\rho_{ab} F^a_i F^b_j = \rho_{ij}. \tag{1.8}$$

because of (1.5). If we put

$$R_{rk} = R^a_{rka}, \quad R^*_{rk} = R_{ra} F^a_k, \quad K_{rk} = K^a_{rka}, \quad K^*_{rk} = K_{ra} F^a_k,$$

we obtain from (1.6), using (1.8)

$$R_{rk} = K_{rk} + (n-4)\rho_{rk} + \varphi F^a_r \rho_{ak} + g_{rk}\rho^a_a + F_{rk} F^a_b \rho^b_a, \tag{1.9}$$

$$R^*_{rk} = K^*_{rk} + (n-4)\rho_{ra} F^a_k + \varphi\rho_{rk} + F_{rk}\rho^a_a + g_{rk} F^a_b \rho^b_a. \tag{1.10}$$

The Ricci tensor K_{ij} of the locally decomposable Riemannin space is a pure tensor, i.e. $K_{ab} F^a_i F^b_j = K_{ij}$, from which $K_{aj} F^a_i = K_{ia} F^a_j$, i.e. K^*_{ij} is a symmetric tensor. The tensor $\rho_{ak} F^a_r$ is a symmetric tensor too, because of (1.8). Therefore, both R_{rk} and R^*_{rk} are symmetric tensors.

We obtain from (1.6)

$$\begin{aligned}R_{irkj} = K_{irkj} + g_{ij}\rho_{rk} - g_{jr}\rho_{ik} + F_{ij} F^a_r \rho_{ak} - F_{jr} F^a_i \rho_{ak} - \\ + g_{ik}\rho_{rj} - F g_{kr}\rho ij + F_{ik}\rho_{aj} + F_{kr} F^a_i \rho_{aj},\end{aligned}$$

and see that

$$R_{rikj} = -R_{irkj}. \tag{1.11}$$

Eliminating ρ_{ij} from (1.6), we obtain [**4**]

$$\begin{aligned}&R^i_{rkj} + b_1[s^{-i}_{rkj} - 2(bR - aR^*) r^i_{rkj} + 2(aR - bR^*) r^i_{akj} F^a_r] - \\ &\quad - a_1[s^{-i}_{akj} F^a_r - 2(bR - aR^*) r^i_{akj} F^a_r + 2(aR - bR^*) r^i_{rkj}] = \\ &= K^i_{rkj} + b_1[s^i_{rkj} - 2(bK - aK^*) r^i_{rkj} + 2(aK - bK^*) r^i_{akj} F^a_r] - \\ &\quad - a_1[s^i_{akj} F^a_r - 2(bK - aK^*) r^i_{akj} F^a_r + 2(aK - bK^*) r^i_{rkj}],\end{aligned}$$

where we have put

$$R = R_a^a, \quad K = K_a^a, \quad R^* = R_a^{*a}, \quad K^* = L_a^{*a},$$

$$\begin{aligned} s_{rkj}^{-i} &= \delta_j^i R_{rk} - g_{rj} R_k^i + F_j^i R_{rk}^* - F_{rj} R_k^{*i} - \\ &-\delta_k^i R_{rj} + g_{rk} R_j^i - F_k^i R_{rj}^* + F_{rk} R_j^{*i}, \\ s_{rkj}^i &= \delta_j^i K_{rk} - g_{rj} K_k^i + F_j^i K_{rk}^* - F_{rj} K_k^{*i} - \\ &-\delta_k^i K_{rj} + g_{rk} K_j^i - F_k^i K_{rj}^* + F_{rk} K_j^{*i} \end{aligned} \tag{1.13}$$

$$r_{rk}^i = \delta_k^i g_{rj} - \delta_j^i g_{rk} + F_k^i F_{rj} - F_j^i F_{rk} \tag{1.14}$$

$$\begin{aligned} a &= \frac{\varphi}{2[\varphi^2 - (n-2)^2]}, \qquad b = \frac{n-2}{2[\varphi - (n-2)^2]}, \\ a_1 &= \frac{\varphi}{\varphi^2 - (n-4)^2}, \qquad b_1 = \frac{n-2}{\varphi^2 - (n-4)^2}. \end{aligned} \tag{1.15}$$

2. Product semisymmetric concircular connection. In this section we consider a special product semisymmetric metric F-connection, namely the connection (1.3) satisfying the condition

$$\rho_{ik} = f g_{ik} + h F_{ik}, \tag{2.1}$$

where f and h are some scalar functions. From (1.6) and (2.1) it results that

$$\begin{aligned} R_{rkj}^i = K_{rkj}^i &+ 2f(\delta_j^i g_{kr} - \delta_k^i g_{jr} + F_j^i F_{kr} - F_k^i F_{jr}) + \\ &+ 2h(\delta_j^i F_{kr} - \delta_k^i F_{jr} + F_j^i g_{kr} - F_k^i g_{jr}). \end{aligned} \tag{2.2}$$

Contracting (2.2) with respect to i and j we obtain

$$R_{rk} = K_{rk} + 2f[(n-2)g_{kr} + \varphi g_{kr}] + 2h((n-2)F_{kr} + \varphi g_{kr}]. \tag{2.3}$$

It follows from (2.3) that

$$R_{rk}^* = K_{rk}^* + 2f[(n-2)F_{kr} + \varphi g_{kr}] + 2h((n-2)g_{kr} + \varphi F_{kr}]. \tag{2.4}$$

Transvecting (2.3) and (2.4) with g^{rk}, we find that

$$\begin{aligned} R - K &= 2f[n(n-2) + \varphi^2] + 4h\varphi(n-1), \\ R^* - K^* &= 4f\varphi(n-1) + 2h[n(n-2) + \varphi^2]. \end{aligned}$$

Since $p > 2$ and $q > 2$, $[n(n-2) + \varphi^2]^2 - 4\varphi^2(n-1)^2 \neq 0$, and the above relations give

$$2f = -\alpha(R-K) - \beta(R^* - K^*), \quad 2h = -\beta(R-K) - \alpha(R^* - K^*), \tag{2.5}$$

where

$$\alpha = -\frac{n(n-2) + \varphi^2}{[n(n-2) + \varphi^2]^2 - 4\varphi^2(n-1)^2}, \qquad \beta = \frac{2\varphi(n-1)}{[n(n-2) + \varphi^2]^2 - 4\varphi^2(n-1)^2}.$$

Substituting (2.5) into (2.2) and taking into account the notation (1.14), we get

$$
\begin{aligned}
(2.7)\qquad & R^i_{rkj} - (\alpha R + \beta R^*) r^i_{rkj} - (\beta R + \alpha R^* r) R^i_{akj} F^a_r = \\
&= K^i_{rkj} - (\alpha K + \beta K^*) r^i_{rkj} - (\beta K + K^*) r^i_{akj} F^o_r.
\end{aligned}
$$

Conversely, we suppose that for (1.3) we have (2.7). Then, substituting $R^i_{rkj} - K^i_{rkj}$ from (2.7) into (1.6), we obtain

$$
\begin{aligned}
&[\alpha(R-K) + \beta(R^* - K^*)] r^i_{rkj} + \beta[(R-K) + \alpha(R^* - K^*) r^i_{akj} F^a_r = \\
&\quad = \delta^i_j \rho_{rk} - \delta^i_k \rho_{rj} - g_{jr} \rho^i_k + g_{kr} \rho^i_j + F^i_j F^a_r \rho_{ak} - F^i_k F^a_r - \\
&\quad F_{jr} g^{ib} F^a_b \rho_{ak} + F_{kr} g^{ib} F^a_b \rho_{aj}.
\end{aligned}
$$

Contracting this with respect to i and k and taking into account (1.8), we get

$$
\begin{aligned}
&(4-n)\rho_{rj} - \varphi F^a_r \rho_{aj} = \\
&= \{(n-2)[\alpha(R-K) + \beta(R^* - K^*)] + \varphi[\beta(R-K) + \alpha(R^* - K^*)] + \rho^a_a\} g_{jr} + \\
&+ \{\varphi[\alpha(R-K) + \beta(R^* - K^*)] + (n-2)[\beta(R-K) + \alpha(R^* - K^*)] + F^a_b \rho^b_a\} F_{ij}
\end{aligned}
$$

or

$$
(2.8)\qquad (4-n)\rho_{rj} - \varphi F^a_r \rho_{aj} = f_1 d_{jr} + h_1 F_{jr},
$$

where we have put

$$
\begin{aligned}
f_1 &= (n-2)[\alpha(R-K) + \beta(R^* - K^*)] + \varphi[\beta(R-K) + \alpha(R^* - K^*)] + \rho^a_a, \\
h_1 &= \varphi[\alpha(R-K) + \beta(R^* - K^*)] + (n-2)[\beta(R-K) + \alpha(R^* - K^*)] + F^a_b \rho^b_a.
\end{aligned}
$$

From (2.8) we obtain $-\varphi \rho_{rj} + (4-n) F^a_r \rho_{aj} = h_1 g_{jr} + \varphi f_1 F_{jr}$. From (2.8) and this last equation we easily find that

$$
[(4-n)^2 - \varphi^2]\rho_{rj} = [(4-n)f_1 + \varphi h_1] g_{jr} + [(4-n)h_1 + \varphi f_1] F_{jr}.
$$

Since $p > 2$ and $g > 2$, $(4-n)f_1 - \varphi^2 \neq 0$, and we can express the preceding relation in the form

$$
\rho_{rj} = \frac{(4-n)f_1 + \varphi h_1}{(4-n)^2 - \varphi^2} g_{jr} + \frac{(4-n)h_1 + \varphi f_1}{(4-n)^2 - \varphi^2} F_{jr}.
$$

This shows that ρ_{rj} has the form (2.1). Therefore, we have

THEOREM 1. *The connection* (1.3) *satisfies* (2.1) *iff* (2.7) *holds.*

The tensor on the right hand side of (2.7) is the product concircular curvature tensor [**3**]. Because of that we introduce the following

Definition. The connection (1.3) satisfying (2.1) is called *PS-concicular (product semisymmetric concircular) connection.*

Contracting (2.7) with respect to i and j, we find

$$\begin{aligned} &R_{rk} + [\varphi(\beta R + \alpha R^*) + (n-2)(\alpha R + \beta R^*)]g_{kr} + \\ &\quad + [(n-2)(\beta R + \alpha R^*) + \varphi(\alpha R + \beta R^*)F_{kr} = \\ &= K_{rk} + [\varphi(\beta K + \alpha K^*) + (n-2)(\alpha K + \beta K^*)]g_{kr} + \\ &\quad + [(n-2)(\beta K + \alpha K^*) + \varphi(\alpha K + \beta K^*)]F_{kr}. \end{aligned} \tag{2.9}$$

Conversely, we suppose that for (1.3), we have (2.9). Then, substituting $R_{rk} - K_{rk}$ from (2.9) into (1.9), we obtain (2.8). Therefore, we have

THEOREM 2. (1.3) *is a PS-concircular connection iff* (2.9) *holds.*

Using the abbreviation

$$A = \alpha R + \beta R^*, \quad B = \beta R + \alpha R^*, \quad P = \alpha K + \beta K^*, \quad Q = \beta K + \alpha K^*, \tag{2.10}$$

we express (2.9) in the form

$$\begin{aligned} &R_{rk} + [\varphi B + (n-2)A]g_{kr} + [(n-2)B + \varphi A]F_{kr} = \\ &= K_{rk} + [\varphi Q + (n-2)P)g_{kr} + [(n-2)Q + \varphi P)F_{kr} \end{aligned}$$

From this, we easily obtain

$$\begin{aligned} &R^*_{rk} = [\varphi B + (n-2)A]F_{kr} + [(n-2)B + \varphi A)g_{kr} = \\ &= K^*_{rk} + [\varphi + (n-2)P]F_{kr} + [(n-2)Q + \varphi P)g_{kr}. \end{aligned}$$

From these two relations it follows that

$$\begin{aligned} &\delta^i_j R_{rk} + [\varphi B + (n-2)A]\delta^i_j g_{kr} + [\varphi A + (n-2)B]\delta^i_j F_{kr} = \\ &= \delta^i_j K_{rk} + [\varphi Q + (n-2)P)\delta^i_j g_{kr} + [\varphi P + (n-2)Q]\delta^i_j F_{kr}, \end{aligned} \tag{2.11}$$

$$\begin{aligned} &F^i_j R_{rk} + [\varphi B + (n-2)A]F^i_j g_{kr} + [\varphi A + (n-2)B]F^i_j F_{kr} = \\ &= F^i_j K_{rk} + [\varphi Q + (n-2)P]F^i_j g_{kr} + [\varphi P + (n-2)Q]F^i_j F_{kr}, \end{aligned} \tag{2.12}$$

$$\begin{aligned} &\delta^i_j R^*_{rk} + [\varphi B + (n-2)A]\delta^i_j F_{kr} + (n-2)B + \varphi A]\delta^i_j g_{kr} = \\ &= \delta^i_j K^*_{rk} + [\varphi Q + (n-2)P]\delta^i_j F_{kr} + [(n-2)Q + \varphi P]\delta^i_j g_{kr}, \end{aligned} \tag{2.13}$$

$$\begin{aligned} &F^i_j R^*_{rk} + [\varphi B + (n-2)A]F^i_j F_{kr} + [(n-2)B + \varphi A]F^i_j g_{kr} = \\ &= F^i_j K^*_{rk} + [\varphi Q + (n-2)P]F^i_j F_{kr} + [(n-2)Q + \varphi P]F^i_j g_{kr}. \end{aligned} \tag{2.14}$$

We multiply (2.11) with $n-2$ and (2.13) with φ and subtract the second from the first. Then, taking into account the notations (1.15), we have

$$\begin{aligned} &A\delta^i_j g_{kr} + B\delta^i_j F_{kr} - P\delta^i_j g_{kr} - Q\delta^i_j F_{kr} = \\ &2b(\delta_j R_{rk} - \delta^i_j K_{rk}) - 2a(R^*_{rk}\delta^i_j - K^*_{rk}\delta^i_j). \end{aligned} \tag{2.15}$$

Now, we multiply (2.12) with φ and (2.14) with $n-2$ and subtract the first from the second. We find

$$\begin{aligned} &BF^i_j g_{kr} + AF^i_j F_{kr} - QF^i_j g_{kr} - PF^i_j F_{kr} = \\ &-2a(F^i_j R_{rk} - F^i_j K_{rk}) + 2b(F^i_j R^*_{rk} - F^i_j K^*_{rk}). \end{aligned} \tag{2.16}$$

On the other hand, we can express (2.7), using the notation (2.10), in the form:

$$\begin{aligned} R^i_{rkj} = K^i_{rkj} &+ (A\delta^i_k g_{jr} + B\delta^i_k F_{jr}) - (P\delta^i_k g_{jr} + Q\delta^i_k F_{jr}) - \\ &- (A\delta^i_j g_{kr} + B\delta^i_j F_{kr}) + (P\delta^i_j g_{kr} + Q\delta^i_j F_{kr}) + \\ &+ (AF^i_k F_{jr} + BF^i_k g_{jr}) - (PF^i_k F_{jr} + QF^i_k g_{jr}) - \\ &- (AF^i_j F_{kr} + BF^i_j g_{kr}) + (PF^i_j F_{kr} + QF^i_j g_{kr}). \end{aligned}$$

Substituting from (2.15) and (2.16), we obtain

$$\begin{aligned} R^i_{rkj} &= -2b(\delta^i_k R_{rj} - \delta^i_j R_{rk} + F^i_k R^*_{rj} - F^i_j R^*_{rk}) + \\ &\quad + 2a(\delta^i_k R^*_{rj} - \delta^i_j R^*_{rk} + F^i_k R_{rj} - F^i_j R_{rk}) = \\ K^i_{rkj} &= -2b(\delta^i_k K_{rj} - \delta^i_j K_{rk} + F^i_k K^*_{rj} - F^i_j K^*_{rk}) + \\ &\quad + 2a(\delta^i_k K^*_{rj} - \delta^i_j K^*_{rk} + F^i_k K_{rj} - F^i_j K_{rk}). \end{aligned}$$

But the tensor on the right-hand side is the product projective curvature tensor [**7**]. Thus we have

THEOREM 3. *The product projective curvature tensor is an invariant of the PS-concircular connection.*

Lowering the index i in the preceding equation and taking into account (1.11), and then raising the index r, we obtain

$$\begin{aligned} R^r_{ikj} &= -2b(g_{ik} R^r_j - g^i_{ij} R^r_k + F_{ik} R^{*r}_j - F_{ij} R^{*r}_k) + \\ &\quad + 2a(g_{ik} R^{*r}_j - g_{ij} R^{*r}_k + F_{ik} R^r_j - F_{ij} R^r_k) = \\ K^r_{rkj} &= -2b(g_{ik} K^r_j - g_{ij} K^r_k + F_{ik} K^{*r}_j - F_{ij} K^{*r}_k) + \\ &\quad + 2a(g^i_{ik} K^{*r}_j - g_{ij} K^{*r}_k + F_{ik} K^r_j - F_{ij} K^r_k), \end{aligned} \tag{2.17}$$

where $R^{*r}_k = R^r_a F^a_k$, $K^{*r}_k = K^r_a F^a_k$.

Conversely, we suppose that for (1.3) we have (2.17). Contracting (2.17) with respect to r and j, we get

$$\begin{aligned} &-[n(n-2) - \varphi^2]R_{ik} + 2\varphi R^*_{ik} + [n(n-2) - \varphi^2]K_{ik} - 2\varphi K^*_{ik} = \\ &= -[(n-2)R - \varphi R^*]g_{ik} - [(n-2)R^* - \varphi R]F_{ik} + \\ &+ [(n-2)K - \varphi K^*]g_{ik} + [(n-2)K^* - \varphi K]F_{ik}. \end{aligned} \tag{2.18}$$

From (2.18) it follows that

$$\begin{aligned} &2\varphi R_{ik} - [n(n-2) - \varphi^2]R^*_{ik} - 2\varphi K_{ik} + [n(n-2) - \varphi^2]K^*_{ik} = \\ &= -[(n-2)R^* - \varphi R]g_{ik} - [(n-2)R - \varphi R^*]F_{ik} + \\ &+ [(n-2)K^* - \varphi K]g_{ik} + [(n-2)K - \varphi K^*]F_{ik}. \end{aligned} \tag{2.19}$$

We multiply (2.18) with $n(n-2)-\varphi^2$ and (2.19) with 2φ and add the obtained relations. Then we have

$$\begin{aligned}R_{ik}-K_{ik}=&-\frac{[n-2)R-\varphi R^*][n(n-2)-\varphi^2]+2\varphi[(n-2)R^*-\varphi R]}{4\varphi^2-[n(n-2)-\varphi^2]^2}g_{ik}-\\&-\frac{[n-2)R^*-\varphi R][n(n-2)-\varphi^2]+2\varphi[(n-2)R-\varphi R^*]}{4\varphi^2-[n(n-2)-\varphi^2]^2}F_{ik}+\\&+\frac{[n-2)K-\varphi K^*][n(n-2)-\varphi^2]+2\varphi[(n-2)K^*-\varphi K]}{4\varphi^2-[n(n-2)-\varphi^2]^2}g_{ik}-\\&-\frac{[n-2)K^*-\varphi K][n(n-2)-\varphi^2]+2\varphi[(n-2)K-\varphi K^*]}{4\varphi^2-[n(n-2)-\varphi^2]^2}F_{ik}.\end{aligned}$$

(Since $p>2$, and $q>2$, $4\varphi^2-[n(n-2)-\varphi^2]^2\neq 0$.)

Substituting $R_{rk}-K_{rk}$ from this relation into (1.9), we obtain a relation of the form (2.8). Therefore, we have

THEOREM 4. *The connection (1.3) is a PS-concircular connection iff (2.17) holds.*

3. Product coharmonic curvature tensor. In this section we consider another special product semisymmetric metric F-connection, namely the connection (1.3) satisfying the conditions

$$\rho_a^a=0,\quad \rho_b^aF_a^b=0. \tag{3.1}$$

Transvecting (1.9) and (1.10) with g_{rk}, we obtain

$$\begin{aligned}R&=K+2(n-2)\rho_a^a+2\varphi F_b^a\rho_a^b,\\R^*&=K^*+2\varphi\rho_a^a+2(n-2)F_b^a\rho_a^b.\end{aligned}\tag{3.2}$$

So, if the connection (1.3) satisfies (3.1), we have $R=K$, $R^*=K^*$ and the relation (1.12) reduces to

$$R^i_{rkj}+b_1\overset{-}{s}{}^i_{rkj}-a_1\overset{-}{s}{}^i_{akj}F^a_r=K^i_{rkj}+b_1s^i_{rkj}-a_1s^i_{akj}F^a_r. \tag{3.3}$$

Conversely, we suppose that for (1.3) we have (3.3). Then, contracting (3.3) with respect to i and j, we get

$$\begin{aligned}&[1+b_1(n-4)-a_1\varphi]R_{rk}+[b_1\varphi-a_1(n-4)]R^*_{rk}+(b_1R-a_1R^*)g_{rk}+\\&\qquad+(b_1R^*-a_1R)F_{rk}=\\&=[1+b_1(n-4)-a_1\varphi]K_{rk}+[b_1\varphi-a_1(n-4)]K^*_{rk}+(b_1R-a_1K^*)g_{rk}+\\&\qquad+(b_1K^*-a_1K)F_{rk}.\end{aligned}$$

Transvecting this equation with g^{rk} and F^{rk}, after some calculation, we obtain

$$[n(n-4)-\varphi^2]R-4R^*=[n(n-4)-\varphi]K-4K^*, \tag{3.4}$$

$$-4R+[n(n-4)-\varphi^2]R^*=-4K+[n(n-4)-\varphi^2]K^*. \tag{3.5}$$

We multiply (3.4) with $n(n-4)-\varphi^2$ and (3.5) with 4 and add the obtained relations. Then we have

$$\{[(n(n-4)-\varphi^2]^2-16\}R=\{[n(n-4)-\varphi^2]^2-16\}K.$$

Since $p>2$ and $q>2$, $[n(n-4)-\varphi^2)^2-16\neq 0$, and therefore $R=K$. Thus (3.2) reduces to

$$(n-2)\rho_a^a+\varphi F_b^a\rho_a^b=0,\quad \varphi\rho_a^a+(n-2)F_b^a\rho_a^b=0,$$

from which $\rho_a^a=0$ and $F_b^a\rho_a^b=0$.

Therefore, we have

THEOREM 5. (1.3) *satisfies the condition* (3.1) *iff* (3.3) *holds.*

The tensor on the right-hand side of (3.3) is analogous to the conharmonic curvature tensor [**2**]. Because of that we introduce the following

Definition. The tensor on the right-hand side of (3.3) is called a *product coharmonic curvatore tensor.*

The connection (1.3) satisfying (3.1) is called a PS-coharmonic connection.

4. Holomorphically semisymmetric connections. The geometrical meaning of the semisymmetric connection was given by E. Bartolotti [**1**] and it consists in the following. Let U and V be two vectors. The vectors $S_{ij}^k u^i v^j$, u^k, v^k are, in the general case, linearly independent. But if

$$S_{ij}^k u^i v^j = pu^k + qv^k \tag{4.1}$$

for every U and V, where p and q are scalars, then S_{ij}^k has the form (0.1), and conversely.

To generalize this property in the case of the locally decomposable Riemannian space, we considered in [**5**] the skew-symmetric tensor S_{ij}^k satisfying the condition $S_{ij}^k u^i F_a^j u^a = pu^k + qF_a^k u^a$ instead of (4.1) and proved the following:

THEOREM. *The skew-symmetric tensor S_{ij}^k satisfies condition (4,1) for every U iff it has the form*

$$\begin{aligned} S_{ij}^k = \delta_j^k S_i - \delta_i^k S_j = F_j^k F_i^a S_a + F_i^k F_j^a S_a + \\ +(\delta_i^a\delta_j^b + F_i^a F_j^b) w_{ab}^k/2, \end{aligned} \tag{4.2}$$

where w_{ab}^k is an arbitrary skew-symmetric tensor.

One connection whose torsion tensor has the form (4.2) is the connection

$$G_{ij}^k = \left\{ {k \atop ij} \right\} + \delta_j^k S_i + \varepsilon g_{ij} S^k - F_j^k F_i^a S_a + \varepsilon F_{ij} F_a^k S^a, \tag{4.3}$$

where $\varepsilon=+1$ or $\varepsilon=-1$. (The case $\varepsilon=-1$ resembles more to the classical semisymmetric metric connection, i.e. to the connection $\left\{ {k \atop ij} \right\} + \delta_j^k S_i - g_{ij}S^k$, but the obtained results hold good for the case $\varepsilon = +1$, too.) This connection is an F-connection, i.e. $\partial F/\partial x^k + G^i_{ka}F^a_j - G^a_{kj}F^i_a = 0$, but is not a metric one.

Definition. The connection c(4.3) is called a *holomorphically semisymmetric* (HS)*-connection.*

The curvature tensor H^i_{rkj} of the connection (4.3) can be expressed in the form [**5**]

$$\begin{aligned} H^i_{rkj} = K^i_{rkj} + \delta^i_r\psi_{jk} - \delta^i_r\psi_{kj} + F^i_rF^a_j\psi_{ak} - F^i_rF^a_k\psi_{aj} + \\ + g_{jr}\psi^i_k - g_{kr}\psi^i_j + F_{jr}F^i_a\psi^a_k - F_{kr}F^i_a\psi^a_j, \end{aligned}$$

where $\psi_{jk} = \varepsilon\nabla_kS_j + S_jS_k + F^a_jF^b_kS_aS_b$, $\psi^i_k = g^{ia}\psi_{ak}$.

As in **1**, we suppose that S_i is locally a gradient and satisfies the condition (1.4). Then

$$\psi_{jk} = \psi_{kj}, \quad F^a_j\psi_{ak} = F^a_k\psi_{aj} \tag{4.4}$$

and the preceding equation reduces to

$$H_{rkj} = K^i_{rkj} + g_{jr}\psi^i_k + g_{kr}\psi^i_j + F_{jr}F^i_a\psi^a_k - F_{kr}F^i_a\psi^a_j. \tag{4.5}$$

If $H^i_{rkj} = H^a_{rka}$, $H^*_{rk} = H_{ak}F^a_r$, $H = H^a_a$, $H^* = H^{*a}_a$, from (4.5) we obtain

$$H_{rk} = K_{rk} + 2\psi_{rk} - g_{kr}\psi^a_a - F_{kr}F^b_a\psi^b_a, \tag{4.6}$$

$$H^*_{rk} = K^*_{rk} + 2\psi_{ak}F^a_rF_{kr}\psi^a_a - g_{kr}F^b_a\psi^a_b. \tag{4.7}$$

Transvecting (4.5) and (4.6) with g^{rk}, we find

$$H - K = (2-n)\psi^a_a - \varphi F^b_a\psi^a_b, \quad H^* - K^* = -\varphi\psi^a_a + (2-n)F^b_a\psi^a_b. \tag{4.8}$$

Taking into account (4.6), (4.7) and (4.8), we can eliminate ψ^i_k from (4.5) and we obtain the relation

$$\begin{aligned} H^i_{rkj} - (g_{jr}H^i_k - g_{kr}H^i_j + F_{jr}H^{*i}_k - F_{kr}H^{*i}_j)/2- \\ - (bH - aH^*)r^i_{rkj} - (bH^* - aH)r^i_{akj}F^a_r = \\ = Krki - (g_{jr}K^i_k - g_{kr}K^i_j + F_{jr}K^{*i}_k - F_{kr}K^{*i}_k)/2- \\ - (bK - QK^*)r^i_{rkj} - (bK^* - aK)r^i_{akj}F^a_r \end{aligned} \tag{4.9}$$

where we have used the notations (1.14) and (1.15).

5. HS-concircular connection. In this section we consider a special HS-connection, namely the connection (4.3) satisfying the condition

$$\psi^i_k = f\delta^i_k + hF^i_k, \tag{5.1}$$

where f and h are some scalar functions.

Substituting (5.1) into (4.5), we find

$$H^i_{rkj} = K^i_{rkj} + fr^i_{rkj} + h^i_{arj}F^a_r. \tag{5.2}$$

Contracting with respect to i and j, we get

$$H_{rk} = K_{rk} + g_{rk}[(2-n)f - \varphi h] + F_{rk}[(2-n)h - \varphi f].$$

Transvecting this with g^{rk} and F^{rk}, we find

$$H - K = [n(2-n) - \varphi^2]f + 2\varphi(1-n)h \tag{5.3}$$

$$H^* - K^* = 2\varphi(1-n)f + [n(2-n) - \varphi^2]h. \tag{5.4}$$

We multiply (5.3) with $n(2-n) - \varphi^2$ and (5.4) with $2(1-n)\varphi$ and subtract the second from the first. Afterward, we multiply (5.3) with $2(1-n)\varphi$ and (5.4) with $n(2-n)-\varphi^2$ and subtract the first from the second. Then, taking into account the notations (2.6), we have

$$f = \alpha(H-K) + \beta(H^* - K^*), \quad h = \beta(H-K) + \alpha(H^* - K^*) \tag{5.5}$$

Substituting (5.5) into (5.2), we find

$$\begin{aligned} &H^i_{rkj} - (\alpha H + \beta H^*)r^i_{rkj} - (\beta H + \alpha H^*)r^i_{akj}F^a_r = \\ &= K^i_{rkj} - (\alpha K + \beta K^*)r^i_{rkj} - (\beta K + \alpha K^*)r^i_{arj}F^a_r. \end{aligned} \tag{5.6}$$

Conversely, we suppose that for (4.3) we have (5.6). Then, substituting $H^i_{rkj} - K^i_{rkj}$ from (5.6) into (4.5), we find

$$\begin{gathered} g_{jr}\psi^i_k - g_{kr}\psi^i_j + F_{jr}F^i_a\psi^a_k - F_{kr}F^i_a\psi^a_j = \\ -(\alpha H + \beta H^* - \alpha K - \beta K^*)r^i_{rkj} + (\beta H + \alpha H^* - \beta K - \alpha K^*)r^i_{akj}F^a_r. \end{gathered}$$

Contracting with respect to i and j we obtain

(5.7)

$$\begin{aligned} &2\psi_{rk} = \\ &g_{rk}\{[\alpha(H-K) + \beta(H^* - K^*)](2-n) - [\beta(H-K) + \alpha(H^* - K^*)]\varphi + \psi^a_a\} + \\ &+ F_{rk}\{-[\alpha(H-K) + \beta(H^* - K^*)]\varphi + [\beta(H-K) + (H^* - K^*)](2-n) + F^b_a\psi^a_b\} \end{aligned}$$

and this is an equation of the form (5.1). Therefore, we have

THEOREM 6. *The connection* (4.3) *satisfies the condition* (5.1) *iff* (5.6) *holds.*

The tensor on the right-hand side of (5.6) being the product concircular curvature tensor, it is reasonable to introduce the following

Definition. The connection (4.3) satisfying (5.1) is called a $HS-concircular$ connection.

Now, we contract (5.6) with respect to i and j and find

$$\begin{aligned}
&H_{rk} + [(n-2)(\alpha H + \beta H^*) + \varphi(\beta H + \alpha H^*)g_{rk} + \\
&\quad + [\varphi(\alpha H + \beta H^*) + (n-2)(\beta H + \alpha H^*)F_{rk} = \\
&= K_{rk} + [(n-2)(\alpha K + \beta K^*) + \varphi(\beta K + \alpha K^*)g_{rk} + \\
&\quad + [\varphi(\alpha K + \beta K^*) + (n-2)(\beta K + \alpha K^*)]F_{rk}.
\end{aligned} \tag{5.8}$$

Conversely we suppose that for the HS-connection (4.3) we have (5.8). Then substituting $H_{rk} - K_{rk}$ from (5.8) into (4.6) we obtain (5.7), i.e. we obtain an equation of the form (5.1). Therefore, we have

THEOREM 7. *An HS-connection ts HS-concircular iff* (5.8) *holds.*

In the same way as in section 2, using the relation (5.8) we obtain

$$\begin{aligned}
&H^i_{rkj} - 2b(\delta^i_k H_{rj} - \delta^i_j H_{rk} + F^i_k H^*_{rj} - F^i_j H^*_{rk}) + \\
&\quad + 2a(\delta^i_k H^*_{rj} - \delta^i_j H^*_{rk} + F^i_k H_{rj} - F^i_j H_{rk}) = \\
&= K^i_{rkj} - 2b(\delta^i_k K_{rj} - \delta^i_j K_{rk} + F^i_k K^*_{rj} - F^i_j K^*_{rk}) + \\
&\quad + 2a(\delta^i_k K^*_{rj} - \delta^i_j K^*_{rk} + F^i_k K_{rj} - F^i_j K_{rk}),
\end{aligned}$$

and therefore we have

THEOREM 8. *The product projective curvature tensor is an invariant of the HS-concircular connection.*

6. Another special HS-connection. In this section we consider HS-connections satisfying the conditions

$$\psi^a_a = 0, \quad \psi^a_b F^b_a = 0. \tag{6.1}$$

Then (4.8) gives $H = K$ and $H^* = K^a st$ and (4.9) reduces to

$$\begin{aligned}
&H^i_{rkj} - (g_{jr} H^i_k - g_{kr} H^i_j + F_{jr} H^{*i}_k - F_{kr} H^{*i}_k)/2 = \\
&= K^i_{rkj} - (g_{jr} K^i_k - g_{kr} K^i_j + F_{jr} K^{*i}_k - F_{kr} K^{*i}_j)/2.
\end{aligned} \tag{6.2}$$

Conversely, we suppose that for the HS-connection (4.3) we have (6.2). Then contracting (6.2) with respect to i and j, we find

$$(H - K)g_{rk} + (H^* - K^*)F_{rk} = O.$$

Transvecting this relation with g^{rk} and F^{rk}, we obtain

$$(H - K)n + (H^* - K^*)\varphi = 0, \quad (H - K)\varphi + (H^* - K^*)n = 0.$$

Consequently $H = K$ and $H^* = K^*$ and (4.8) reduces to

$$(2-n)\psi^a_a - \varphi\psi^a_b F^b_a = O, \quad -\varphi\psi^a_a + (2-n)F^b_a\psi^a_b = 0,$$

from which (6.1) follows. Therefore we have

Theorem 9. *An HS-connection satisfies the condition* (6.1) *iff* (6.1) *holds.*

7. Remark concerning HS-connections. Lowering the index i in (4.5) we have

$$H_{irkj} = 2K_{irkj} + g_{jr}\psi_{ik} - g_{kr}\psi_{ij} + F_{jr}F_{ia}\psi^a_k - F_{kr}F_{ia}\psi^a_j.$$

From this we obtain

$$\begin{aligned} H_{irkj} - H_{rikj} = 2K_{irkj} &- g_{ji}\psi_{rk} + g_{ki}\psi_{rj} - F_{ji}F^a_r\psi_{ak} + F_{ki}F^a_r\psi_{aj} + \\ &+ g_{jr}\psi_{ik} - g_{kr}\psi_{ij} + F_{jr}F^a_i\psi_{ak} - F_{kr}F^a_i\psi_{aj}. \end{aligned}$$

Let us introduce the following notation

$$L_{irkj} = (H_{irkj} - H_{rikj})/2, \quad \rho_{rk} = -\psi_{rk}/2.$$

Then we express the preceding relation in the form

$$\begin{aligned} L_{irkj} = K_{irkj} &+ g_{ji}\rho_{rk} - g_{ki}\rho_{rj} + F_{ji}F^a_r\rho_{ak} - F_{ki}F^a_r\rho_{aj} - \\ &- g_{jr}\rho_{ik} + g_{kr}\rho_{ij} - F_{jr}F^a_i\rho_{ak} + F_{kr}F^a_i\rho_{aj}. \end{aligned}$$

or, raising the index i, in the form

$$\begin{aligned} L^i_{rkj} = K^i_{rkj} &+ \delta^i_j\rho_{rk} - \delta^i_k\rho_{rj} + F^i_jF^a_r\rho_{ak} - F_{ki}F^a_r\rho_{aj} - \\ &- g_{jr}\rho^i_k + g_{kr}\rho^i_j - F_{jr}F^{ia}\rho_{ak} + F_{kr}F^{ia}\rho_{aj}. \end{aligned}$$

The right-hand side of this relation has the same form as the right-hand side of (1.6). Therefore, all conclusions of 2 and 3 can be repeated for H S-connections end the tensor L^i_{rkj}.

REFERENCES

[1] E. Bartolotti, *Sulla geometria della varieta a connesione affine*, Ann. di Mat.,1930 (4), VIII, 53–101.

[2] Y. Ishii, *On conharmonic transformations*, Tensor **7** (1957), 73–86.

[3] V. Petrović, *Product-concircular curvature tensor*, Publ. Inst. Math. (Beograd) (N. S.) **25**(**39**) (1979), 131–137.

[4] M. Prvanović, *Product semisymmetric connections of the locally decomposable Riemannian space*, Bull. Acad. Serbe Sci. Arts Cl. Sci. Math. Natur. **64** (1979), 17–27.

[5] M. Prvanović, *Holomorphically semisymmetric connections*, Univ. u Novom Sadu, Zb. Rad. Prirod. Mat. Fak. **9** (1979), 91–99.

[6] E. Stavre, *On the S-concircular and S-coharmonic connections*, Tensor (N. S.) **38** (1982), 103–108.

[7] S. Tachibana, *Some theorems on locally product Riemannion space*, Tôhoku Math. J. **12** (1960), 281–292.

[8] K. Yano, *Differential Geometry on Complex and Almost Complex Spaces*, Pergamon Press, Oxford, 1965.

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