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ORIENTATION OF ABSOLUTE SPACE S^n

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Abstract. We consider one way of founding the orientation of absolute space S^n. Briefly, we show how the orientation can be introduced by using the first two groups of axioms: axioms of incidence and axioms of order. In [**1**] the orientation of E^2 is found in a similar way, but using an analytical method which needs all five groups of axioms.

1. Oriented Simplexes and Chaias

An oriented simplex is a simplex whose vertices are ordered. We shall consider only oriented simplexes and for the sake of simplicity we shall call them simplexes. By the simplex $A_0A_1 \dots A_n$ we shall mean the simplex whose first vertex is A_0, whose second vertex is A_1 and so on.

Simplexes $A_0A_1 \dots A^n$ and $B_0B_1 \dots B_n$ are connected if $A_i = B_{i-1}$, $i = 1, \dots, n$. A chain is a finite sequence of simplexes such that any two consecutive members are connected. A chain is closed if its initial and terminal members coincide. By the chain $A_0A_1 \dots A^n$, $(m > n)$ we shall mean the sequence of simplexes $A_iA_{i+1} \dots A_{i+n}$, $i = 0, 1, \dots, m - n$. By the closed chain $A_0A_1 \dots A^n$ $(m > n)$ we shall mean the sequence of simplexes $A_iA_{i+1} \dots A_{i+n}$ $i = 0, 1, \dots, m$, where $A_{m+j} = A_j$ for $j = 0, 1, \dots, n$. Chain C connects simplex S with simplex S' if it starts at S and if it terminates at $\mathbf{S}'$.

Theorem 1. *For any two simplexes S and S' there exists a chain which connects S with S'.*

Proof. Let $S = A_0A_1 \dots A^n$ and $S' = B_0B_1 \dots B^n$. Furthermore,

- let C_1 be a point which doesn't lie on the hyperplane $A_0A_1 \dots A^n$, and which is distinct from the point B_0;

- let C_2 be a point which doesn't lie on the hyperplane $A_1A_2 \dots A^nC_1$, and which doesn't lie on the lines C_1B_0 and B_0B_1;

- let C_3 be a point which doesn't lie on the hyperplane $A_3A_4 \dots A^nC_1C_2$, and which doesn't lie on the planes $C_1C_2B_0$, $C_2B_0B_1$ and $B_0B_1B_2$;...

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- let C_n be a point which doesn't lie. on the hyperplanes $A_nC_1\dots C_{n-1}$, $C_1C_2\dots, C_{n-1}B_0, \dots C_{n-1}B_0\dots B_{n-2}$ and $B_0B_1\dots B_{n-1}$. It is obvious that the chain $C = A_0A_1\dots A_nC_1\dots C_nB_0B_1\dots B_n$ connects the simplex S with the simplex S'. ■

2. Parity of Chains

A couple of connected simplexes $A_0A_1\dots A_n$ and $A_1\dots A_nA_{n+1}$ is anti-oriented if the vertices A_0 and A_{n+1} lie on the same side of the hyperplane determined by the common side $A_1\dots A_n$ for n odd, and if the vertices A_0 and A_{n+1} lie on the opposite sides of the hyperplane determined by the common side $A_1\dots A_n$ for n even. A parity of a chain is the parity of the number of anti-oriented couples of consecutive members of that chain.

THEOREM 2. *Closed chains are even.*

Proof for $n = 1$. Suppose that the closed chain $A_0A_1\dots A_{m-1}$ is given. The points $A_0, A_1, \dots, A_{m-1}$ can be enumerated by the intogers $a_0, a_1, \dots, a_{m-1}$ such that $(a_i - a_j)(a_j - a_k) > 0$ if and only if A_j lies between A_i and A_k. The couple of segments A_iA_{i+1} and $A_{i+1}A_{i+2}$ is anti-oriented if and only if $(a_i - a_{i+1})(a_{i+1} - a_{i+2}) < 0$. Since

$$\prod_{i=1}^{m-1}(a_i - a_{i+1})(a_{i+1} - a_{i+2}) = \prod_{i=1}^{m-1}(ai - a_{i+1})^2 > 0,$$

the number of anti-oriented couples of consecutive members of the given chain is even, i.e. the given chain is even. ■

Let H be a hyperplane and let A and B be two points which don't lie on H. Let us define $a(A,H,B)$ and $b(A,H,B)$ as

$$a(A,H,B) = \begin{cases} 1, & A,B \overset{\cdot\cdot}{-} H \\ -1, & A,B \div H \end{cases}$$
$$b(A,H,B) = \begin{cases} 1, & A,B \div H \\ -1, & A,B \overset{\cdot\cdot}{-} H \end{cases}$$

These two functions have the following two properties:
a) If points A, B and C don't lie on the hyperplane H, then

$$a(A,H,B)a(B,H,C)a(C,H,A) = 1,$$
$$b(A,H,B)b(B,H,C)b(C,H,A) = -1.$$

b) If three points A, B and C and the plane P of codimension 2 determine three distinct hyperplanes, then

$$a(A,BP,C)a(B,CP,A)a(C,AP,B) = -1,$$
$$b(A,BP,C)b(B,CP,A)b(C,AP,B) = 1.$$

LEMMA. *Let A_0, $A_1, \dots,$ A_m be points in S^n $(n>1)$. Then there exist points A'_0, $A'_1, \dots,$ A'_m in S^n such that no $n+1$ of them lie on one hyperplane and such that*

$$B(A'_i, A'_{k_1}, A'_{k_2}, \dots, A'_{k_n}, A'_j) \Leftrightarrow B(A_i, A_{k_1}, A_{k_2}, \dots, A_{k_n}, A_j) \qquad (*)$$

provided the points, $A_{k_1}, A_{k_2}, \dots, A_{k_n}$ determine a unique hyperplane and the points A_i and A_j don't lie on that hyperplane.

Proof. Let l_0 be a line passing through A_0 which doesn't lie on any hyperplane determined by n points among A_0, $A_1, \dots,$ A_m. Let A'_0 be a point lying on l_0 such that the segment $]A_0, A'_0]$ doesn't have a common point with any considered hyperplane. The points A'_0, $A_1, \dots,$ A_m satisfy the given condition. If $i=0$, the condition $(*)$ is satisfied because, according to the property a) of the function b,

$$b(A'_0, A_{k_1}A_{k_2}\dots A_{k_n}, A_j) = -b(A'_0, A_{k_1}A_{k_2}\dots A_{k_n}, A_0)b(A_0, A_{k_1}A_{k_2}\dots A_{k_n}, A_j)$$

and, according to the way point A'_0 is chosen,

$$b(A'_0, A_{k_1}A_{k_2}\dots A_{k_n}, A_0) = -1.$$

The case $j=0$ can be considered in the same way. If $k_1=0$, the condition $(*)$ is satisfied, because, according to the property b) of the function b,

$$b(A_i, A'_0A_{k_2}\dots A_{k_n}, A_j) = b(A'_0, A_iA_{k_2}\dots A_{k_n}, A_j)b(A_i, A_jA_{k_2}\dots A_{k_n}, A'_0),$$
$$b(A_iA_0A_{k_2}\dots A_{k_n}, A_j) = b(A_0, A_iA_{k_2}\dots A_{k_n}, A_j)b(A_i, A_jA_{k_2}\dots A_{k_n}, A_0),$$

and the equalities

$$b(A'_0, A_iA_{k_2}\dots A_{k_n}, A_j) = b(A_0, A_iA_{k_2}\dots A_{k_n}, A_j),$$
$$b(A_i, A_jA_{k_2}\dots A_{k_n}, A'_0) = b(A_i, A_jA_{k_2}\dots A_{k_n}, A_0),$$

have been already proved. The cases $k_2=0, \dots, k_n=0$ can be considered in the same way.

Repeating the same procedure with the points $A_1, \dots, A_m$, we shall get the points $A'_0, A'_1, \dots, A'_m$ which satisfy all of the needed conditions. ∎

Proof of Theorem 2 for odd $n>1$. The closed chain $A_0A_1\dots A_{m-1}$ is even if and only if -1 occurs an even number of times among the numbers $b(Ai, A_{i+1}A_{i+2}\dots A_{i+n}, A_{i+n+1})$, $i=0,1,\dots,m-1$, i.e., if and only if

$$\prod_{i=0}^{m-1} b(A_i, A_{i+1}A_{i+2}\dots A_{i+n}, A_{i+n+1}) = 1.$$

We shall prove our statement by induction on m.

For $m=n+1$ the statement is valid, because all of the numbers $b(A_i, A_{i+1}$ $A_{i+2}\dots A_{i+n}, A_{i+n+1})$ are equal to -1 and m is even.

Let us suppose that the statement is valid for integer $m>n$. Suppose that the closed chain $A_0A_1\dots A_m$, is given. By the previous lemma, we can suppose that no $n+1$ among the points A_0, $A_1, \ldots$, A_m belong to one hyperplane. By the induction hypothesis, the statement is valid for the closed chain $A_0 A_1 \ldots, A_{m-1}$, and therefore $P = 1$, where

$$P = \left(\prod_{i=0}^{m-n-2} b(A_i, A_{i+1} \ldots A_{i+n}, A_{i+n+1})\right) \cdot b(A_{m-n-1}, A_{m-n} \ldots A_{m-1}, A_0) \cdot$$
$$\cdot \prod_{j=0}^{n-1} b(A_{m-n+j}, A_{m-n+j+1} \ldots A_{m-1} A_0 \ldots A_j, A_{j+1}).$$

It remains to show that $Q = 1$, where

$$Q = \left(\prod_{i=0}^{m-n-2} b(A_i, A_{i+1} \ldots A_{i+n}, A_{i+n+1})\right) \cdot b(A_{m-n-1}, A_{m-n} \ldots A_{m-1}, A_m) \cdot$$
$$\cdot \left(\prod_{j=0}^{n-1} b(A_{m-n+j}, A_{m-n+j+1} \ldots A_m A_0 \ldots A_{j-1}, A_j)\right) \cdot b(A_m, A_0 \ldots A_{n-1}, A_n).$$

Using the properties b) and a) of the function b we get

$$Q = \left(\prod_{i=0}^{m-n-2} b(A_i, A_{i+1} \ldots A_{i+n}, A_{i+n+1})\right) \cdot b(A_{m-n-1}, A_{m-n} \ldots A_{m-1}, A_m) \cdot$$
$$\cdot \prod_{j=0}^{n-1} [b(A_m, A_{m-n+j} \ldots A_{m-1} A_0 \ldots A_{j-1}, A_j) \cdot$$
$$\cdot b(A_{m-n+j}, A_{m-n+j+1} \ldots A_{m-1} A_0 \ldots A_j, A_m)] \cdot b(A_{m-n-1}, A_{m-n} \ldots A_{m-1}, A_m),$$

$$Q = \left(\prod_{i=0}^{m-n-2} b(A_i, A_{i+1} \ldots A_{i+n}, A_{i+n+1})\right) \cdot b(A_{m-n-1}, A_{m-n} \ldots A_{m-1}, A_m) \cdot$$
$$\cdot b(A_m, A_{m-n} \ldots A_{m-1} A_0) \cdot \prod_{j=0}^{n-1} [b(A_m, A_{m-n+j+1} \ldots A_{m-1} A_0 \ldots A_j, A_{j+1}) \cdot$$
$$\cdot b(A_{m-n+j}, A_{m-n+j+1} \ldots A_{m-1} A_0 \ldots A_j, A_m)],$$

$$Q = \left(\prod_{i=0}^{m-n-2} b(A_i, A_{i+1} \ldots A_{i+n}, A_{i+n+1})\right) \cdot (-b(A_{m-n-1}, A_{m-n} \ldots A_{m-1}, A_0)) \cdot$$
$$\cdot \prod_{j=0}^{n-1} (-b(A_{m-n+j}, A_{m-n+j+1} \ldots A_{m-1} A_0 \ldots A_j, A_{j+1})),$$

$$Q = \left(\prod_{i=0}^{m-n-2} b(A_i, A_{i+1} \ldots A_{i+n}, A_{i+n+1})\right) \cdot b(A_{m-n-1}, A_{m-n} \ldots A_{m-1}, A_0) \cdot$$
$$\cdot \prod_{j=0}^{n-1} b(A_{m-n+j}, A_{m-n+j+1} \ldots A_{m-1} A_0 \ldots A_j, A_{j+1}),$$

$$Q = P = 1. \quad \blacksquare$$

Proof of Theorem 2 for n even. The closed chain $A_0A_1\dots A_{m-1}$ is even if and only if -1 occurs an even number of times among the numbers $a(A_i, A_{i+1}A_{i+2}\dots A_{i+n}, A_{i+n+1})$, $i=0,1,\dots,m-1$, i.e., if and only if

$$\prod_{i=0}^{m-1} a(A_i, A_{i+1}A_{i+2}\dots A_{i+n}, A_{i+n+1}) = 1.$$

We shall prove our statement by induction on m.

For $m=n+1$ the statement is valid, because all of the numbers $a(A_i, A_{i+1}A_{i+2}\dots A_{i+n}, A_{i+n+1})$ are equal to 1.

Let us suppose that the statement is valid for integer $m>n$. Suppose that the closed chain $A_0A_1\dots A_m$ is given. By the previous lemma, we can suppose that no $n+1$ among the points $A_0, A_1, \dots, A_m$ belong to one hyperplane. By the induction hypothesis, the statement is valid for the closed chain $A_0A_1\dots A_{m-1}$, and therefore $P=1$, where

$$P = \left(\prod_{i=0}^{m-n-2} a(A_i, A_{i+1}\dots A_{i+n}, A_{i+n+1})\right) \cdot a(A_{m-n-1}, A_{m-n}\dots A_{m-1}, A_0) \cdot \\ \cdot \prod_{j=0}^{n-1} a(A_{m-n+j}, A_{m-n+j+1}\dots A_{m-1}A_0\dots A_j, A_{j+1}).$$

It remains to show that $Q=1$, where

$$Q = \left(\prod_{i=0}^{m-n-2} a(A_i, A_{i+1}\dots A_{i+n}, A_{i+n+1})\right) \cdot a(A_{m-n-1}, A_{m-n}\dots A_{m-1}, A_0) \cdot \\ \cdot \left(\prod_{j=0}^{n-1} a(A_{m-n+j}, A_{m-n+j+1}\dots A_{m-1}A_0\dots A_j, A_{j+1})\right) \cdot a(A_m, A_0\dots A_{n-1}, A_n).$$

Using the properties b) and a) of the function a we get

$$Q = \left(\prod_{i=0}^{m-n-2} a(A_i, A_{i+1}\dots A_{i+n}, A_{i+n+1})\right) \cdot a(A_{m-n-1}, A_{m-n}\dots A_{m-1}, A_m) \cdot \\ \cdot \left(\prod_{j=0}^{n-1} [-a(A_m, A_{m-n+j}\dots A_{m-1}A_0\dots A_{j-1}, A_j) \cdot \right. \\ \left. \cdot\, a(A_{m-n+j}, A_{m-n+j+1}\dots A_{m-1}A_0\dots A_j, A_m)]\right) \cdot a(A_m, A_0\dots A_{n-1}, A_n),$$

$$Q = \left(\prod_{i=0}^{m-n-2} a(A_i, A_{i+1}\dots A_{i+n}, A_{i+n+1})\right) \cdot a(A_{m-n-1}, A_{m-n}\dots A_{m-1}, A_m) \cdot \\ \cdot\, a(A_m, A_{m-n}\dots A_{m-1}A_0) \cdot \prod_{j=0}^{n-1} [a(A_m, A_{m-n+j+1}\dots A_{m-1}A_0\dots A_j, A_{j+1}) \cdot \\ \cdot\, a(A_{m-n+j}, A_{m-n+j+1}\dots A_{m-1}A_0\dots A_j, A_m)],$$

$$Q = \left(\prod_{i=0}^{m-n-2} a(A_i, A_{i+1} \dots A_{i+n}, A_{i+n+1}) \right) \cdot a(A_{m-n-1}, A_{m-n} \dots A_{m-1}, A_0) \cdot$$
$$\cdot \prod_{j=0}^{n-1} a(A_{m-n+j}, A_{m-n+j+1} \dots A_{m-1} A_0 \dots A_j, A_{j+1}),$$
$$Q = P = 1. \ \blacksquare$$

THEOREM 3. *If the chains C and C' have the same origin and the same end, they have the same purity.*

Proof. Let C'' be a chain which connects the common end of C and C' with their common origin. If we extend C by C'' we get a closed chain. We thus conclude that the total number of anti-oriented couples of consecutive members of C and C'' is even. Therefore C and C'' have the same parity. In the same way we conclude that C' and C'' have the same parity. It follows that C and C' have the same parity. ■

3. Orientations

Simplex S has the same orientation as simplex S', or briefly $S \rightrightarrows S'$, if each chain which connects them is even. Simplex S has the opposite orientation to simplex S', or briefly $S \rightleftarrows S'$, if each chain which connects them is odd.

THEOREM 4. *Relation $\rightrightarrows$ is an equivalence relation which defines the partition of the family of simplexes into two equivalence classes.*

Proof. This relation is reflexive because each closed chain is even.

Let $S \rightrightarrows S'$ and $S' \rightrightarrows S''$. Let C be a chain which connects S' with S'', and let C'' be the chain which is the extension of C by C''. The chain C'' is even, because chains C and C' are even. Therefore $S \rightrightarrows S''$. We conclude that the relation $\rightrightarrows$ is transitive.

Let $S \rightrightarrows S'$. Let C be a chain which connects S with S', let C' be a chain which connects S' with S, and let C'' be the chain which is the extension of C by C'. Chain C' is even, because chains C and C'' are even. Therefore $S' \rightrightarrows S$. We conclude that the relation $\rightrightarrows$ is symmetric.

Let $S \rightleftarrows S'$ and $S' \rightleftarrows S$. Let C be a chain which connects S with S', let C' be a chain which connects S' with S, and let C'' be the extension of C by C'. The chain C'' is even, because chains C and C'' are odd. Therefore $S \rightrightarrows S''$. ■

An orientation of n-dimensional absolute space S^n is any equivalence class with respect to the relation $\rightrightarrows$. There are two opposite orientations of space S^n.

THEOREM 5. *Let p be a permutation of numbers $0, 1, \dots, n$. Simplex $A_{p_0} A_{p_1} \dots A_{p_n}$ has the same orientation as simplex $A_0 A_1 \dots A_n$ if and only if the permutation p is even.*

Proof. First, let us prove that the statement is valid for the transposition (0,1). Let A be any inner point of the $n-1$-dimensional simplex $A1 \dots A_n$. Let us consider the chain $A_1A_0A_2 \dots A_nAA_0A_1 \dots A_n$ which connects the simplex $A_1A_0A_2 \dots A_n$ with the simplex $A_0A_1 \dots A_n$. The number of anti-oriented couples of consecutive members of this chain is 3 if n is odd, and it is $n-1$ if n is even. Therefore, this chain is odd, and so the simplex $A_1A_0A_2 \dots A_n$ has the opposite orientation to the simplex $A_0A_1 \dots A_n$.

Now, let us prove that the statement is valid for the cyclic permutation $(0, 1, \dots, n)$. It has the same parity as n. Let us consider the chain $A_1A_2 \dots A_nA_0$ $A_1 \dots A_n$ which connects the simplex $A_1 \dots A_nA_0$ with the simplex $A_0A_1 \dots A_n$. The number of anti-oriented couples of consecutive members of this chain is 0 if n is even, and it is n for n odd. Therefore, this chain has the same parity as n. So, the simplex $A_1 \dots A_nA_0$ has the same orientation as the simplex $A_0A_1 \dots A_n$ if and only if the cyclic permutation $(0, 1, \dots, n)$ is even.

Finally, let us prove that if the statement is valid for permutations p and q, then it is valid for their composition qp. Let S be an arbitrary simplex, let S' be the simplex obtained from S by reordering its vertices by the permutation p, and let S'' be the simplex obtained from S' by reordering its vertices by the permutation q. Simplex S'' arises from simplex S by reordering its vertices by the permutation qp. We shall consider four cases:

1. If the permutations p and q are even, the permutation qp is even. From $S \rightrightarrows S'$ and $S' \rightrightarrows S''$ it follows that $S \rightrightarrows S''$.

2. If the permutation p is even and if the permutation q is odd the permutation qp is odd. From $S \rightrightarrows S'$ and $S' \rightleftarrows S''$ it follows that $S \rightleftarrows S''$.

3. If the permutation p is odd and if the permutation q is even, the permutation qp is odd. From $S \rightleftarrows S'$ and $S' \rightleftarrows S''$ it follows that $S \rightleftarrows S''$.

4. If the permutations p and q are odd, the permutation qp is even. From $S \rightleftarrows S'$ and $S' \rightleftarrows S''$ it follows that $S \rightrightarrows S''$.

It remains to remark that the transposition (0,1) and the cyclic permutation $(0, 1, \dots, n)$ generate all permutations of numbers $0, 1, \dots, n$). ■

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