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ON A NEW SUBCLASS OF ANALYTIC P-VALENT FUNCTIONS

Shigeyoshi Owa

Abstract. There are many classes of analytic and p-valent functions in the unit disk U. N. S. Sohi studied a class $S_p(\alpha)$ of analytic and p-valent functions

$$f(z) = z^p + \sum{}^{*)} a_{p+n} z^{p+n}, \qquad (p \in N)$$

in the unit disk U satisfying the condition

$$|f'(z)/pz^{p-1} - \alpha| < \alpha, \qquad (z \in U)$$

for $\alpha > 1/2$. In this paper, we consider a new subclass $S_{p,k}(\alpha)$ of analytic and p-valent functions

$$f(z) = z^p + \sum a_{p+n} z^{p+n}, \qquad (p \in N)$$

in the unit disk U satisfying the condition

$$\left| \frac{\Gamma(p+1-k) D_z^k(z)}{\Gamma(p+1) z^{p-k}} \right| < \alpha, \qquad (z \in U)$$

for $0 < k < 1$, $\alpha > 1/2$ and $p \in N$, where $D_z^k f(z)$ means the fractional derivative of order k of $f(z)$. It is the purpose of this paper to show a distortion theorem, the coefficient estimates and a convolution theorem for the class $S_{p,k}(\alpha)$. Further we give a theorem about convex set of functions in the class $S_{p,k}(\alpha)$.

1. Introduction

There are many definitions of the fractional calculus, that is, the fractional derivatives and the fractional integrals. In 1978, S. Owa [**8**] showed the following definitions for the fractional calculus.

Definition 1. The fractional integral of order k is defined by

$$D_z^{-k} f(z) = \frac{1}{\Gamma(k)} \int_0^z \frac{f(\zeta) d\zeta}{(z-\zeta)^{1-k}},$$

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$^{*)}\sum$ stands for $\displaystyle\sum_{n=1}^{\infty}$ unless stated otherwise.

where $k > 0$, $f(z)$ is an analytic function in a simply connected region of the z-plane containing the origin and the multiplicity of $(z-\zeta)^{k-1}$ is removed by requiring $\log(z-\zeta)$ to be real when $(z-\zeta) > 0$.

Definition 2. The fractional derivative of order k is defined by

$$D_z^k f(z) = \frac{1}{\Gamma(1-k)} \frac{1}{dz} \int_0^1 \frac{f(\zeta)d\zeta}{(z-\zeta)^k},$$

where $O < k1$, $f(z)$ is an analytic function in a simply connected region of the z-plane containing the origin and the multiplicity of $(z-\zeta)^{-k}$ is removed by requiring $\log(z-\zeta)$ to be real when $(z-\zeta) > 0$.

Definition 3. Under the hypotheses of Definition 2, the fractional derivative of order $(n+k)$ is defined by $D_z^{n+k} f(z) = d^n D_z^k f(z)/dz^n$ where $0 < k < 1$ and $n \in N \cup \{0\}$.

Remark 1. For other definitions of the fractional calculus, see R. P. Agarwal [**1**], W. A. Al-Salam [**2**], K. Nishimoto [**6**], T. J. Osler [**7**], B. Ross [**10**] and M. Saigo [**11**].

Let $S_p(\alpha)$ denote the class of functions $f(z) = z^p + \sum a_{p+n} z^{p+n}$, $(p \in N)$ which are analytic and p-valent in the unit disk $U = \{|z| < 1\}$ and which satisfy the condition $|f'(z)/pz^{p-1} - \alpha| < \alpha$, $(z \in U)$ for $\alpha > 1/2$. This class $Sp(\alpha)$ was studied by N. S. Sohi [**12**]. In particular, R. M. Goel [**3**], [**4**] studied the class $S_1(\alpha)$.

Further let $S_{p,k}(\alpha)$ denote the class of functions $f(z) = z^p + \sum a_{p+n} z^{p+n}$, $(p \in N)$, which are analytic and p-valent in the unit disk U and which satisfy the condition

$$\left| \frac{\Gamma(p+1-k) D_z^k f(z)}{\Gamma(p+1) z^{p-k}} - \alpha \right|, \qquad (z \in U)$$

for $0 < k < 1$, $\alpha > 1/2$ and $p \in N$.

Remark 2. R. M. Goel and N. S. Sohi [**5**], H. M. Srivastava and S. Owa [**13**] studied other subclasses of analytic and p-valent functions in the unit disk U.

2. Distortion theorem

THEOREM 1. *Let the function $f(z) = z^p + \sum a_{p+n} z^{p+n}$, $(p \in N)$ be in the class $S_{p,k}(\alpha)$. Then we have*

$$\frac{\Gamma(p+1)|z|^{p-k}(1-|z|))}{\Gamma(p+1-k)(1-A|z|)} \le |D_z^k f(z)| \le \frac{\Gamma(p+1)|z|^{p-k}(1+|z|)}{\Gamma(p+1-k)(1+A|z|)}$$

for $z \in U$, where $A = 1/\alpha - 1$. The estimates are sharp.

Proof. Let

$$g(z) = \frac{\Gamma(p+1-k) D_z^k f(z)}{\alpha \Gamma(p+1) z^{p-k}} - 1,$$

then $g(z)$ has modulus at most 1 in the unit disk U and $g(0) = 1/\alpha - 1$. Again let $h(z) = (g(z) - g(0))/(1 - g(0)g(z))$, so that $h(z)$ vanishes at the origin and $|hz)| < 1$ for $z \in U$. Therefore, by Schwarz's lemma, we have $h(z) = z\varphi(z)$, where $\varphi(z)$ is an analytic function in the unit disk U and satisfies $|\varphi(z)| \leq 1$ for $z \in U$. Consequently we obtain

$$\frac{\Gamma(p+1-k)D_z^k f(z)}{\Gamma(p+1)z^{p-k}} = \frac{1+h(z)}{1+Ah(z)} = \frac{1+z\varphi(z)}{1+Az\varphi(z)},$$

where $A = 1/\alpha - 1$. After a simple computation, we get

$$\left|\frac{\Gamma(p+1-k)D_z^k f(z)}{\Gamma(p+1)z^{p-k}} - \frac{1-A|z|^2}{1-A^2|z|^2}\right| \leq \frac{(1-A)|z|}{1-A^2|z|^2}$$

which shows that

$$\frac{\Gamma(p+1-k)|z|^{p-k}(1-|z|)}{\Gamma(p+1-k)(1-A|z|)} \leq D_z^k f(z) \leq \frac{\Gamma(p+1-k)|z|^{p-k}(1+|z|)}{\Gamma(p+1-k)(1-A|z|)}$$

for $z \in U$. Finally, choosing the function $f(z)$ such that

$$D_z^k f(z) = \frac{\Gamma(p+1-k)|z|^{p-k}(1+z)}{\Gamma(p+1-k)(1+Az)},$$

we can show that the estimates of this theorem are sharp.

Corollary 1. *Under the hypotheses of Theorem 1, we have*

$$\arg\left|\frac{\Gamma(p+1-k)D_z^k f(z)}{\Gamma(p+1)z^{p-k}}\right| \leq \sin^{-1}\left(\frac{(1-A)|z|}{(1-A)|z|^2}\right),$$

where $A = 1/\alpha - 1$.

Proof. Since $f(z) \in S_{p,k}(\alpha)$, in view of Theorem 1, we have

$$\left|\frac{\Gamma(p+1-k)D_z^k f(z)}{\Gamma(p+1)z^{p-k}} - \frac{1-A|z|^2}{1-A^2|z|^2}\right| \leq \frac{(1-A)|z|}{1-A^2|z|^2},$$

where $A = 1/\alpha - 1$. This gives the corollary.

3. Coefficient estimates

Theorem 2. *Let the function* $f(z) = z^p + \sum a_{p+n}z^{p+n}$, $(p \in N)$ *be in the class* $S_{p,k}(\alpha)$. *Then we have*

$$|a_{p+n}| \leq \frac{(2-1/\alpha)\Gamma(p+1)\Gamma(p+n+1-k)}{\Gamma p+n+1\Gamma(p+1-k)}$$

for $n \leq 2$. *The estimates are sharp.*

Proof. Since $f(z) \in S_{p,k}(\alpha)$, in view of Theorem 1, we have

$$\frac{\Gamma(p+1-k)D_z^k f(z)}{\Gamma(p+1)z^{p-k}} - \frac{1+h(z)}{1+Ah(z)},$$

where $h(z)$ vanishes at the origin, $|h(z)| < 1$ for $z \in U$ and $A = 1/\alpha - 1$. Consequently we obtain

$$\left\{(1-A) - A\sum \frac{\Gamma(p+n+1)\Gamma(p+1-k)}{\Gamma(p+1)(p+n+1-k)} a_{p+n}z^n\right\} h(z) =$$
$$= \sum \frac{\Gamma(p+n+1)\Gamma(p+1-k)}{\Gamma(p+1)\Gamma(p+n+1-k)} a_{p+n}z^n.$$

Hence we can write

$$\left\{(1-A) - A\sum_{n=1}^{m} \frac{\Gamma(p+n+1)\Gamma(p+1-k)}{\Gamma(p+1)(p+n+1-k)} a_{p+n}z^n\right\} h(z) =$$
$$= \sum_{n=1}^{m+1} \frac{\Gamma(p+n+1)\Gamma(p+1-k)}{\Gamma(p+1)\Gamma(p+n+1-k)} a_{p+n}z^n + \sum_{n=m+1}^{\infty} c_n z^n$$

c_n being some complex numbers. Since $|h(z)| < 1$ for $z \in U$, by using Parseval's identity, we have

$$\sum_{n=1}^{m+1} \frac{\Gamma(p+n+1)^2\Gamma(p+1-k)^2}{\Gamma(p+1)^2\Gamma(p+n+1-k)^2} |a_{p+n}|^2|z|^{2n} + \sum_{n=m+1}^{\infty} |c_n|^2|z|^{2n} \leq$$
$$\leq (1_A)^2 + A^2 \sum_{n=1}^{m} \frac{\Gamma(p+n+1)^2\Gamma(P+1-k)^2}{\Gamma(p+1)^2\Gamma(p+n+1-k)^2} |a_{p+n}|^2|z|^{2n},$$

that is,

$$\sum_{n=1}^{m+1} \frac{\Gamma(p+n+1)^2\Gamma(p+1-k)^2}{\Gamma(p+1)^2\Gamma(p+n+1-k)^2} |a_{p+n}|^2|z|^{2n} \leq$$
$$\leq (1-A)^2 + A^2 \sum_{n=1}^{m} \frac{\Gamma(p+n+1)^2\Gamma(p+1-k)^2}{\Gamma(p+1)^2\Gamma(p+n+1-k)^2} |a_{p+n}|^2|z|^{2n}.$$

On letting $|z| \to 1$,

$$\sum_{n=1}^{m+1} \frac{\Gamma(p+n+1)^2\Gamma(p+1-k)^2}{\Gamma(p+1)^2\Gamma(p+n+1-k)^2} |a_{p+n}|^2 \leq$$
$$\leq (1-A)^2 + A^2 \sum_{n=1}^{m} \frac{\Gamma(p+n+1)^2\Gamma(p+1-k)^2}{\Gamma(p+1)^2\Gamma(p+n+1-k)^2} |a_{p+n}|^2.$$

Accordingly we get

$$\frac{\Gamma(p+m+1)^2\Gamma(p+1-k)^2}{\Gamma(p+1)^2\Gamma(p+m+1-k)^2}|a_{p+n}|^2 \leq$$
$$\leq (1-A)^2-(1-A^2)\sum_{n=1}^{m}\frac{\Gamma(p+n+1)^2\Gamma(p+1-k)^2}{\Gamma(p+1)^2\Gamma(p+n+1-k)^2}|a_{p+n}|^2 \leq (2-1/\alpha)^2.$$

Thus we can show that

$$|a_{p+n}| \leq \frac{(2-1/\alpha)\Gamma(p+1)\Gamma(p+n+1-k)}{\Gamma(p+n+1)\Gamma(p+1-k)}$$

for $n \geq 2$. Considering the function $f(z) \in S_{p,k}(\alpha)$ which has the expansion

$$f(z) = z^p + \frac{(2-1/\alpha)\Gamma(p+1)\Gamma(p+n+1-k)}{\Gamma(p+n+1)\Gamma(p+1-k)}z^{p+n} + \dots$$

for $z \in U$, we can see that the estimate is sharp.

4. Convolution for the class $S_{p,k}(\alpha)$

THEOREPM 3. *Let the functions* $f(z) = z^p + \sum a_{p+n}z^{p+n}$, $(p \in N)$ *and* $g(z) = z^p + \sum b_{p+n}z^{p+n}$ $(p \in N)$ *be in the same class* $S_{p,k}(\alpha)$. *Then the function*

$$F(z) = z^p + \frac{1}{2}\sum\frac{\Gamma(p+n+1)\Gamma(p+1-k)}{\Gamma(p+1)\Gamma(p+n+1-k)}a_{p+n}b_{p+n}z^{p+n}$$

is also in the class $S_{p,k}(\alpha)$.

Proof. Since $f(z) \in S_{p,k}(\alpha)$ and $g(z) \in S_{p,k}(\alpha)$, we have

$$\left|\frac{\Gamma(p+1-k)D_z^k f(z)}{\Gamma(p+1)z^{p-k}} - \alpha\right| < \alpha,\ \ (z \in U) \qquad \left|\frac{\Gamma(p+1)D_z^k g(z)}{\Gamma(p+1)z^{p-k}} - \alpha\right| < \alpha,\ \ (z \in U).$$

Now, it is well-known that if the function

$$w(z) = \sum_{n=0}^{\emptyset} d_n z^n$$

is analytic in the unit disk U and $|w(z)| \leq M$, then

$$\sum_{n=0}^{\infty}|d_n|^2 \leq M^2.$$

Applying the above estimate to the functions

$$\frac{\Gamma(p+1-k)D_z^k f(z)}{\Gamma(p+1)z^{p-k}} - \alpha \quad \text{and} \quad \frac{\Gamma(p+1-k)D_z^k g(z)}{\Gamma(p+1)z^{p-k}} - \alpha,$$

we obtain

$$\sum \frac{\Gamma(p+n+1)^2\Gamma(p+1-k)^2}{\Gamma(p+1)^2\Gamma(p+n+1-k)^2}|a_{p+n}|^2 \le 2\alpha - 1$$
$$\sum \frac{\Gamma(p+n+1)^2\Gamma(p+1-k)^2}{\Gamma(p+1)^2\Gamma(p+n+1-k)^2}|b_{p+n}|^2 \le 2\alpha - 1$$

Hence we have

$$\begin{aligned}
&\left|\frac{\Gamma(p+1-k)D_z^k f(z)}{\Gamma(p+1)z^{p-k}} - \alpha\right|^2 = \\
&= \left|(1-\alpha) + \frac{1}{2}\sum \frac{\Gamma(p+n+1)^2\Gamma(p+1-k)^2}{\Gamma(p+1)^2\Gamma(p+n+1-k)^2} a_{p+n} b_{p+n} z^n\right|^2 \\
&\le (1-\alpha)^2 + (1-\alpha)\sum \frac{\Gamma(p+n+1)^2\Gamma(p+1-k)^2}{\Gamma(p+1)^2\Gamma(p+n+1-k)^2}|a_{p+n}||b_{p+n}||z|^n \\
&+ \frac{1}{4}\left\{\sum \frac{\Gamma(p+n+1)^2\Gamma(p+1-k)^2}{\Gamma(p+1)^2\Gamma(p+n+1-k)^2}|a_{p+n}||b_{p+n}||z|^n\right\}^2 \\
&\le (1-\alpha)^2 + (1-\alpha)\sum \frac{\Gamma(p+n+1)^2\Gamma(p+1-k)^2}{\Gamma(p+1)^2\Gamma(p+n+1-k)^2}|a_{p+n}||b_{p+n}| \\
&+ \frac{1}{4}\left\{\sum \frac{\Gamma(p+n+1)^2\Gamma(p+1-k)^2}{\Gamma(p+1)^2\Gamma(p+n+1-k)^2}|a_{p+n}||b_{p+n}|\right\}^2 \\
&\le (1-\alpha)^2 + (1-\alpha)\left\{\sum \frac{\Gamma(p+n+1)^2\Gamma(p+1-k)^2}{\Gamma(p+1)^2\Gamma(p+n+1-k)^2}|a_{p+n}|^2\right\}^{1/2} \\
&\times \left\{\sum \frac{\Gamma(p+n+1)^2\Gamma(p+1-k)^2}{\Gamma(p+1)^2\Gamma(p+n+1-k)^2}|b_{p+n}|^2\right\}^{1/2} \\
&+ \frac{1}{4}\left\{\sum \frac{\Gamma(p+n+1)^2\Gamma(p+1-k)^2}{\Gamma(p+1)^2\Gamma(p+n+1-k)^2}|a_{p+n}|^2\right\}^2 \\
&\times \left\{\sum \frac{\Gamma(p+n+1)^2\Gamma(p+1-k)^2}{\Gamma(p+1)^2\Gamma(p+n+1-k)^2}|b_{p+n}|^2\right\}^2 \\
&\le (1-\alpha)^2 + (1-\alpha)(2\alpha-1) + (2\alpha-1)^2/4 < \alpha^2
\end{aligned}$$

because $\alpha > 1/2$. This proves that $F(z)$ belongs to the class $S_{p,k}(\alpha)$.

5. Convex set of functions

THEOREM 4. *Let the functions $f(z) = z^p + \sum a_{p+n}z^{p+n}$, $(p \in N)$ and $g(z) = z^p + \sum b_{p+n}z^{p+n}$, $(p \in N)$ be in the same class $S_{p,k}(\alpha)$. Then $\lambda f(z) + (1-\lambda)g(z)$ is also in the class $S_{p,k}(\alpha)$, where $0 \le \lambda 1$.*

Proof. Since $f(z) \in S_{p,k}(\alpha)$ and $g(z) \in S_{p,k}(\alpha)$, we have

$$\left|\frac{\Gamma(p+1-k)D_z^k f(z)}{\alpha\Gamma(p+1)z^{p-k}} - 1\right| < 1, \ \ (z \in U) \quad \left|\frac{\Gamma(p+1-k)D_z^k g(z)}{\alpha\Gamma(p+1)z^{p-k}} - 1\right| < 1, \ \ (z \in U).$$

Consequently we obtain

$$\left|\frac{\lambda\Gamma(p+1-k)D_z^k f(z)+(1-\lambda)\Gamma(p+1-k)D_z^k g(z)}{\alpha\Gamma(p+1)z^{p-k}}-1\right|$$
$$\leq\left|\frac{\Gamma(p+1-k)D_z^k f(z)}{\alpha\Gamma(p+1)z^{p-k}}-1\right|+(1-\lambda)\left|\frac{\Gamma(p+1-k)D_z^k g(z)}{\alpha\Gamma(p+1)z^{p-k}}-1\right|\leq 1$$

for $z\in U$. This completes the proof of the theorem.

Remark 3. Letting $k\to 1$ in our theorems, we have the results obtained by N. S. Sohi [**13**].

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Department of Mathematics
Kinki University
Osaka, Japan

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