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INTUITIONISTIC DOUBLE NEGATION AS A NECESSITY OPERATOR

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Abstract. An intuitionistic propositional modal logic in which we have a necessity operator equivalent to intuitionistic double negation is proved sound and complete with respect to Kripke-style models with two relations, one intuitionistic and the other modal. It is shown how the holding of formulae characteristic for this logic is equivalent to conditions for the relations of the models.

0. Introduction. In this paper we shall investigate an intuitionistic propositional modal logic in which we have a modal operator $\Box$ which is equivalent to intuitionistic double negation. In this logic $\Box A \leftrightarrow \neg\neg A$ is a theorem, but whereas $\neg\neg$ is divisible into two negations, $\Box$ is a single indivisible operator. We shall prove the soundness and completeness of this logic with respect to Kripke-style models with two accessibility relations, one intuitionistic and the other modal. This type of models was investigated in [**1**] and [**3**], and we shall presuppose an acquaintance with these papers, as well as an acquaintance with Kripke models for intuitionistic propositional logic (see, for example, [**4**] and Kripke models for normal modal logics based on classical propositional logic (see, for example, [**2**]). The present paper is an attempt to apply the techniques of [**1**] and [**3**] to an intuitionistic modal operator with a natural interpretation.

1. The syntax of $Hdn\Box$. The language $L\Box$ is the language of propositional modal logic with denumerably many propositional variables, for which we use the schemata p, q, r, $p_1, \dots$, and the connectives $\to$, $\wedge$, $\vee$, $\neg$ and $\Box$ (the connective $\leftrightarrow$ is defined as usual in terms of $\to$ and $\wedge$, and in formulae $\wedge$ and $\vee$ bind more strongly than $\to$ and $\leftrightarrow$). As schemata for formulae we use A, $B, \dots$, $A_1, \dots$, and as schemata for sets of formulae we use capital Greek letters.

The system $Hdn\Box$ ("H" stands for "Heyting" and "dn" for "double negation") is an extension of the Heyting propositional calculus in $L\Box$ (axiomatized in a standard way with modus ponens as in section 2 of [**1**]) with

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$dn1.$	$\Box(A \to B) \to (\Box A \to \Box B)$
$dn2.$	$A \to \Box A$
$dn3.$	$\Box(((A \to B) \to A) \to A)$
$dn4.$	$\neg\Box\neg(A \to A).$

It is not difficult to show that the system obtained by replacing $dn1 - dn4$ by

$dn0.$ $\Box A \leftrightarrow \neg\neg A$

has the same theorems as $Hdn\Box$. (We note that in order to prove $\Box A \to \neg\neg A$ in $Hdn\Box$ we don't need $dn3$, whereas in order to prove $\neg\neg A \to \Box A$ we don't need $dn4$.) Using $dnl - dn4$ is however more suitable than using $dn0$ when we want to connect $Hdn\Box$ with the models we shall give for this system.

Since $Hdn\Box$ is closed under replacement of equivalent formulae, as can easily be shown, the schema $dn0$ guarantees that $\Box$ in $Hdn\Box$ stands for intuitionistic double negation. (It is of course trivial to show that $Hdn\Box$ is a conservative extension of the Heyting propositional calculus in $L\Box$ without $\Box$.) It is also not difficult to show that the rule

$$\frac{A \to B}{\Box A \to \Box B}$$

is derivable in $Hdn\Box$, and that the schemata $\Box A \wedge \Box B \to \Box(A \wedge B)$ and $\Box(A \to A)$ are provable in $Hdn\Box$. Since this rule and this schemata are characteristic for the system $HK\Box$ — the minimal normal intuitionistic modal logic with the necessity operator $\Box$ (see [**1**]) – we can connect intuitionistic double negation with $\Box$. In other words, $Hdn\Box$ is an extension of $HK\Box$, and since it is closed under substitution for propositional variables, it is a normal extension. (In fact, $Hdn\Box$ is an extension of $HD\Box$, since $dn4$ is the schema $\Box D$ of [**3**].) On the other hand, we shall not connect intuitionistic double negation with the possibility operator $\Diamond$, because $\Diamond(A \vee B) \to \Diamond A \vee \Diamond B$, which is one of the schemata characteristic for $HK\Diamond$ – the minimal normal intuitionistic modal logic with $\Diamond$ (see [**1**]) – does not hold when $\Diamond$ is interpreted as intuitionistic double negation.

Note that in $Hdn\Box$ we can prove $\Box A \leftrightarrow \neg\Box\neg\Box A$ which goes some way towards explaining why intuitively $\Box$ in $Hdn\Box$ has some features of possibility as well as some features of necessity.

(If $dn0$ is added to the Heyting predicate calculus, the unprovable Double Negation Shift formula, related to Kuroda's conjecture, $\forall x\neg\neg A \to \neg\neg\forall x A$ becomes equivalent to the Barcan formula.)

2. $Hdn\Box$ **models.** First we summarize some terminology and results of [**1**]. A $H\Box$ frame is $\langle X, R_I, R_M\rangle$ where $X \neq \emptyset$, $R_I \subseteq X^2$ is reflexive and transitive, $R_M \subseteq X^2$ and $R_I R_M \subseteq R_M R_I$. The variables $x, y, z, t, u, v, x_1, \dots$ range over X. A $H\Box$ model is $\langle X, R_I, R_M, V\rangle$ where $\langle X, R_I, R_M\rangle$ is a $H\Box$ frame and V, called a valuation, is a mapping from the set of propositional variables of $L\Box$ to the power set of X such that for every p, $\forall x, y(xR_I y \Rightarrow (x \in V(p) \Rightarrow y \in V(p)))$. The relation $\models$ in $x \models A$ is defined as usual, except that for $\to$ and $\neg$ it involves R_I, whereas $x \models \Box A \Leftrightarrow_{df} \forall y(xR_M y \Rightarrow y \models A)$. A formula A holds in a model $\langle X, R_I, R_M, V\rangle$

iff $\forall x \in X.\ x \models A$; A holds in a frame Fr ($Fr \models A$) iff A holds in every model with this frame; and A is valid iff A holds in every frame. A $H\Box$ frame (model) is condensed iff $R_I R_M = R_M$, and it is strictly condensed iff $R_I R_M = R_M R_I = R_M$. The system $HK\Box$ is sound and complete with respect to $H\Box$ models (condensed $H\Box$ models, strictly condensed $H\Box$ models).

Next we give the following definition: $\langle X, R_I, R_M\rangle$ ($\langle X, R_I, R_M, V\rangle$) is a *$Hdn\Box$ frame (model)* iff

(1) it is a $H\Box$ frame (model)

(2) $R_M \subseteq R_I$

(3) $\forall x, y(xR_My \Rightarrow \forall z(yR_Iz \Rightarrow zR_Iy))$

(4) $\forall x \exists y. xR_My$ (i.e., R_M is serial).

We shall show that $Hdn\Box$ is sound and complete with respect to $Hdn\Box$ models.

3. Equivalence of $dn2-dn4$ with conditions on $H\Box$ frames. Before our soundness and completeness proof we shall give some lemmata about $dn2 - dn4$. We shall show that the holding of these formulae in $H\Box$ frames Fr is equivalent to specific conditions concerning the relations of the frames, viz. the conditions (2) – (4) in the definition of $Hdn\Box$ frames. (These lemmata are analogous to those in section 4 of [**3**].) No such condition corresponds to $dn1$, since this schema is provable in $HK\Box$.

Lemma 1. $Fr \models A \to \Box A \Leftrightarrow R_M \subseteq R_I$.

Proof. ($\Rightarrow$) Suppose for some x and y, xR_My *and not* xR_Iy. Let $\forall u(u \models p \Leftrightarrow$ *not* $uR_Iy)$. By Lemma 4 (ii) of [**3**] there is a valuation such that this is satisfied. With this valuation $x \models p$ and $y \not\models p$. From xR_My *and* $y \not\models p$ we obtain $x \not\models \Box p$. Hence, $x \not\models p \to \Box p$.

($\Leftarrow$) From Intuitionistic Heredity (Lemma 2 of [**1**]) we have $x \models A \Rightarrow (xR_Iy \Rightarrow y \models A)$, which together with $R_M \subseteq R_I$ gives $x \models A \Rightarrow (xR_My \Rightarrow y \models A)$, i.e., $x \models A \Rightarrow x \models \Box A$. From this $Fr \models A \to \Box A$ follows. q.e.d.

Lemma 2. $Fr \models \Box(((A \to B) \to A) \to A) \Leftrightarrow \forall x, y(xR_My \Rightarrow \forall z(yR_Iz \Rightarrow zR_Iy))$.

Proof. ($\Rightarrow$) Suppose for some x, y and z, xR_My *and* yR_Iz *and not* zR_Iy. Let $\forall u(u \models p \Leftrightarrow$ *not* $uR_Iy)$ and $\forall u.\, u \not\models q$. It is easy to check that there is a valuation such that this is satisfied (we again appeal to Lemma 4 (ii) of [**3**] for the case with p). With this valuation we have

$vR_Iy \Rightarrow vR_Iz$, since yR_Iz

yR_Iv *and* $vR_Iy \Rightarrow vR_Iz$ *and not* zR_Iy , since *not* zR_Iy

yR_Iv *and* $vR_Iy \Rightarrow \exists t(vR_It$ *and not* $tR_Iy)$

yR_Iv *and* $\forall t(vR_It \Rightarrow tR_Iy) \Rightarrow$ *not* vR_Iy

yR_Iv *and* $\forall t(vR_It$ *and* $t \models p \Rightarrow t \models q) \Rightarrow v \models q$

$y \models (p \to q) \to p$.

On the other hand we have $y \not\models p$, and hence $y \not\models ((p \to q) \to p) \to p$. So, $x \not\models \Box(((p \to q) \to p) \to p)$.

($\Leftarrow$) Suppose the right-hand side of the Lemma, and suppose that for some x, $x \not\models \Box(((A \to B) \to A) \to A)$. This implies that there is a y such that xR_My *and* $y \not\models ((A \to B) \to A) \to A$, and this implies that there is a t such that yR_It *and* $t \not\models (A \to B) \to A$ *and* $\not\models A$. It follows that $t \not\models A \to B$, and hence there is a z such that tR_Iz *and* $z \models A$ *and* $z \not\models B$. From yR_It *and* tR_Iz it follows that yR_Iz, which with the right-hand side of the Lemma and xR_My implies zR_Iy. But then by Intuitionistic Heredity we obtain $y \models A$ and $t \models A$, which is a contradiction. q.e.d.

If we assume that R_I is not only reflexive and transitive, but that it is a full partial ordering (as we may well do), then $\forall x, y(xR_My \Rightarrow zR_Iy))$ becomes $\forall x, y(xR_My \Rightarrow \mathit{not}(\exists z \neq y)yR_Iz)$ i.e., xR_My only if y is maximal with respect to R_I.

We have already proved in Lemma 5 of [**3**] that

$$Fr \models \neg\Box\neg(A \to A) \Leftrightarrow R_M \text{ is serial.}$$

Consider now what conditions would be equivalent (in the sense of the lemmata above) to $dn2 - dn4$ in Kripke frames $Fr_c = \langle X, R\rangle$ appropriate for normal modal logics based on classical propositional logic. For $dn4$ we would obtain the same as above, and $dn3$ as well as $dn1$, is provable in the modal logic K, and hence it has no corresponding condition. For $dn2$ we would have

$$Fr_c \models A \to \Box A \Leftrightarrow \forall x, y(xRy \Rightarrow x = y).$$

The condition on the right-hand side of this equivalence is the converse of the reflexivity of R — this reflexivity could be written as $\forall x, y(x = y \Rightarrow xRy)$. It is well known that $Fr_c \models \Box A \to A \Leftrightarrow R$ is reflexive. With classical logic $dn2$ and $dn4$ entail $\Box A \to A$, and hence also $\Box A \leftrightarrow A$, for we have

$$\dfrac{\dfrac{dn4}{\Box A \to \neg\Box\neg A} \qquad \dfrac{\dfrac{dn2}{\neg\Box\neg A \to \neg\neg A}}{\neg\Box\neg A \to A}(*)}{\Box A \to A}$$

So, the condition equivalent to the conjunction of $dn2$ and $dn4$ would be that R is the identity relation. With intuitionistic logic the step marked with $(*)$ in the derivation above is blocked, and $\Box A \to A$ is not a theorem of $Hdn\Box$.

4. Soundness and completeness of $Hdn\Box$. Before proceeding with our soundness and completeness proof we review briefly some more terminology from [**1**]. A set of formulae Γ is nice iff Γ is consistent, deductively closed (i.e., $\{A | \Gamma \vdash A\} \subseteq \Gamma$, where $\vdash$ is the usual relation of deductibility from hypotheses using *only modus ponens*) and it has the disjunction property (i.e.,

$A \vee B \in \Gamma \Rightarrow A \in \Gamma$ *or* $B \in \Gamma$). In the canonical $Hdn\Box$ frame $\langle X^c, R_I^c, R_M^c\rangle$ (canonical $Hdn\Box$ model $\langle X^c, R_I^c, R_M^c, V^c\rangle$) X^c is the set of all sets of formulae which are nice with respect to $Hdn\Box$, $\Gamma R_I^c \Delta$ is defined as $\Gamma \subseteq \Delta$, and $\Gamma R_M^c \Delta$ as $\Gamma_\Box \subseteq \Delta$, where $\Gamma_\Box = \{A | \Box A \in \Gamma\}$ ($V^c(p)$ is $\{\Gamma | p \in \Gamma\}$). Then we prove the following lemma.

LEMMA 3. *The canonical $Hdn\Box$ frame (model) is a $Hdn\Box$ frame (model).*

Proof. First we show that in the canonical $Hdn\Box$ frame $R_M^c \subseteq R_I^c$. Suppose $\Gamma_\Box \subseteq \Delta$, and let $A \in \Gamma$. Then since Γ is nice, $\Box A \in \Gamma$, by using $A \to \Box A$. Hence, $A \in \Gamma_\Box$, and by using $\Gamma_\Box \subseteq \Delta$ we get $A \in \Delta$. So $\Gamma \subseteq \Delta$.

Next we show that in the canonical $Hdn\Box$ frame $\forall\Gamma, \Delta(\Gamma R_M^c \Delta \Rightarrow \forall\Theta(\Delta R_I^c \Theta \Rightarrow \Theta R_I^c \Delta))$. Suppose $\Gamma_\Box \subseteq \Delta$. Since $\Box(((A \to B) \to A) \to A) \in \Gamma$, $((A \to B) \to A) \to A \in \Delta$. Then we easily obtain that $A \vee \neg A \in \Delta$ and that the nice set Δ is maximal consistent.

Finally, to show that in the canonical $Hdn\Box$ frame R_M^c is serial we proceed as for Lemma 13 (i) of [**3**].

Using these facts and Lemma 7 of [**1**], which guarantees that the canonical $Hdn\Box$ frame (model) is a $H\Box$ frame (model), we obtain the Lemma. q.e.d.

Then we can show our soundness and completeness theorem.

THEOREM 1. $\vdash_{Hdn\Box} A \Leftrightarrow$ *for every $Hdn\Box$ frame Fr, Fr $\models A$.*

Proof. ($\Rightarrow$) Soundness follows from the ($\Leftarrow$) parts of the lemmata of section 3 and from the soundness of $HK\Box$ with respect to $H\Box$ frames.

($\Leftarrow$) For completeness we proceed quite analogously to what we had for the completeness part of Theorem 1 of [**1**], or of Theorem 1 of [**3**]. (It is quite easy to prove that the set of theorems of $Hdn\Box$ has the disjunction property.) q.e.d.

We can also give another soundness and completeness theorem, which follows from the soundness part of Theorem 1 and from $R_M^c R_I^c = R_M^c$ (cf. Theorem 2 of [**1**]).

THEOREM 2. $\vdash_{Hdn\Box} A \Leftrightarrow$ *for every condensed $Hdn\Box$ frame Fr, Fr $\models A$*
$\Leftrightarrow$ *for every strictly condensed $Hdn\Box$ frame*
Fr, Fr $\models A$.

With easy examples it is possible to show that strictly condensed $Hdn\Box$ frames form a proper subclass of condensed $Hdn\Box$ frames, which form a proper subclass of the class of all $Hdn\Box$ frames.

It is not difficult to check that in the definition of strictly condensed $Hdn\Box$ frames the conditions $R_I R_M \subseteq R_M R_I$, (2), (3) and $R_I R_M = R_M R_I = R_M$, can all be replaced by the condition

$$\forall x, y(xR_M y \Leftrightarrow xR_I y \text{ and} \forall z(yR_I z \Rightarrow zR_I y))$$

yielding the same class of frames. So in these frames R_M is definable in terms of R_I. Now, if in the definition of $Hdn\Box$ frames we require that R_I is not only reflexive and transitive, but a partial ordering, our soundness and completeness results still hold (just note that in the canonical $Hdn\Box$ frame $\subseteq$ is a partial-ordering relation). However, in that case all $Hdn\Box$ frames are strictly condensed (just show $R_M R_I \subseteq R_M$). Hence, we have shown $Hdn\Box$ sound and complete with respect to partially ordered frames where for any x there is a maximal element y above x, xR_My means that y is a maximal element above x, and $x \models \Box A$ means that A holds in all maximal elements above x.[1] (These frames are analogous to the frames with respect to which the Heyting predicate calculus with the formula $\forall x\neg\neg A \to \neg\neg\forall xA$ is proved sound and complete in [**5**], pp. 41, 57–58.)

We shall conclude this paper with the following question. Let S be the system whose theorems are those theorems of $Hdn\Box$ in which $\neg$ does not occur. The system S extends the positive fragment of the Heyting propositional calculus with intuitionistic double negation, but not with negation. This system is decidable, and it should be sound and complete with respect to $Hdn\Box$ models. We leave open the question whether S can be naturally axiomatized.[2]

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[1] This completeness result was first proved by Dr Milan Božić. The connection with our strictly condensed $Hdn\Box$ frames was noted later.

[2] Added in proof: This problem was solved in the meantime by Dr Milan Božić. It is treated in a paper in this issue of *Publications de l'Institut Mathématique* and also in an abstract in the Bulletin of the Section of Logic **12** (1983) No 3, pp. 99–104.

After these texts went into print I learned that an answer to the question above was also announced by Dr Vladimir Sotirov on the Seventh International Wittgenstein Symposium (Kirchberg am Wechsel, 1982, Abstracts, p. 58).