PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 35 (49), 1984, pp. 3–14

NEGATIVE MODAL OPERATORS IN INTUITIONISTIC LOGIC

Kosta Došen

Abstract. Modal operators which correspond to impossibility and non-necessity are investigated in systems analogous to the modal logic K which are based on the Heyting propositional calculus. Soundness and completeness are proved with respect to Kripke-style models with two accessibility relations, one intuitionistic and the other modal. A system where impossibility is equivalent to intuitionistic negation is also proved sound and complete with respect to specific classes of models with two relations. It is shown how the holding of formulae characteristic for this system is equivalent to conditions for the relations of the models.

0. Introduction. Let X be a set of "worlds" and R an "accessibility relation" on X. If $x, y \in X$ and A is a formula, the conditions for the holding of $\Box A$ and $\Diamond A$ at x are

$$x \models \Box A \Leftrightarrow \forall y(xRy \Rightarrow y \models A), \quad x \models \Diamond A \Leftrightarrow \exists y(xRy \ \textit{and}\ y \models A).$$

Now suppose we have introduced the modal operators $\Diamond'$ and $\Box'$ for which we give the following conditions, where the right-hand sides are obtained by negating the right-hand sides of the conditions above,

$$x \models \Box' A \Leftrightarrow \exists y(xRy \ \textit{and}\ y \not\models A), \quad x \models \Diamond' A \Leftrightarrow \forall y(xRy \Rightarrow y \not\models A).$$

With classical propositional logic, $\Diamond'$ and $\Box'$ do not introduce anything particularly new, since $\Diamond' A$ and $\Box' A$ are definable as $\neg\Diamond A$ (or $\Box\neg A$) and $\neg\Box A$ (or $\Diamond\neg A$) respectively. (Though, of course, they can also be treated as primitive. In fact, $\Diamond'$ was a primitive of the *Survey of Symbolic Logic* of C.I. Lewis; see [**6**], pp. 126–128, where $\Box'$ is mentioned too.) However, with intuitionistic propositional logic $\Diamond'$ and $\Box'$ need not be definable in this way anymore. It is our purpose in this paper to investigate the operators $\Diamond'$ and $\Box'$ in propositional modal calculuses based on the Heyting propositional calculus.

In the first part we shall consider a system called $HK\Diamond'$ with $\Diamond'$ primitive, and we shall prove $HK\Diamond'$ sound and complete with respect to Kripke-style models with two accessibility relations, one intuitionistic and the other modal. In these

AMS Subject Classification (1980): Primary 03B45

models the modal relation will be as general as possible, and hence $HK\Diamond'$ will be in the same position as the modal logic K based on the classical propositional calculus. Models with two relations for intuitionistic modal logics with $\Box$ or $\Diamond$ primitive were investigated in [**1**] and [**3**], and we shall assume that the reader is acquainted with these two papers – in particular with the first. (We also assume some familiarity with Kripke models for intuitionistic propositional logic, for which the reader may consult [**4**], and with Kripke models for modal logics based on classical propositional logic, for which the reader may consult [**2**].)

In the second part we shall deal with the system $HK\Box'$ with $\Box'$ primitive, which is in a position analogous to that of $HK\Diamond'$. This system can be proved sound and complete with respect to other classes of models with two relations. The system with $\Diamond'$ bears a certain primacy not only because $\Diamond'$ is intuitively more significant (it corresponds to impossibility, whereas $\Box'$ corresponds to non-necessity), but also because $\Diamond'$ is analogous to negation when we compare the condition given for $x \models \Diamond' A$ with the condition for $x \models \neg A$. In fact, $\Diamond'$ in $HK\Diamond'$ can be conceived as a kind of negation, weaker than intuitionistic negation.

In the third part of this paper we shall consider briefly a system with $\Diamond'$ primitive where $\Diamond'$ will be equivalent to intuitionistic negation. We shall show this system sound and complete with respect to specific classes of models of the type we have mentioned.

1. Review of some terminology and results. In order to make this paper a little bit more self-contained we shall review briefly some terminology and results of [**1**].

The language $L\Box$ is the language of propositional modal logic with denumerably many propositional variables, for which we use the schemata $p, q, r, p_1, \ldots$, and the connectives $\to, \wedge, \vee, \neg$ and $\Box$ (the connective $\leftrightarrow$ is defined as usual in terms of $\to$ and $\wedge$, and in formulae $\wedge$ and $\vee$ bind more strongly than $\to$ and $\leftrightarrow$). As schemata for formulae we use $A, B, \ldots, A_1, \ldots$, and as schemata for sets of formulae we use capital Greek letters.

The system $HK\Box$ is an extension of the Heyting propositional calculus in $L\Box$ (axiomatized in a standard way with *modus ponens*) with

$$\Box 1.\ \Box A \wedge \Box B \to \Box(A \wedge B) \qquad\qquad R\Box.\ \frac{A \to B}{\Box A \to \Box B}.$$
$$\Box 2.\ \Box(A \to A)$$

The relation $\vdash$ in $\Phi \vdash A$ is defined as the usual relation of deducibility from hypotheses *using only modus ponens*. The expression $Cl(\Phi)$ shall stand for $\{A \mid \Phi \vdash A\}$.

A $H\Box$ frame is $\langle X, R_I, R_M\rangle$ where $X \neq \emptyset$, $R_I \subseteq X^2$ is reflexive and transitive, $R_M \subseteq X^2$ and $R_I R_M \subseteq R_M R_I$. The variables $x, y, z, t, u, v, x_1 \ldots$ range over X. A $H\Box$ model is $\langle X, R_I, R_M, V\rangle$ where $\langle X, R_I, R_M\rangle$ is a $H\Box$ frame and V, called a valuation, is a mapping from the set of propositional variables of $L\Box$ to the power set of X such that for every p, $\forall x, y(xR_I y \Rightarrow (x \in V(p) \Rightarrow y \in V(p)))$. The relation $\models$ in $x \models A$ is defined as usual, except that for $\to$ and $\neg$ it involves R_I whereas for $\Box$ it involves R_M. A formula A holds in a model $\langle X, R_I, R_M, V\rangle$ iff $\forall x \in X$, $x \models A$; A holds in a frame $Fr(Fr \models A)$ iff A holds in every model with this frame; and A is valid iff A holds in every frame. A $H\Box$ frame (model) is condensed iff $R_I R_M = R_M$, and it is strictly condensed iff $R_I R_M = R_M R_I = R_M$.

The system $HK\Box$ is sound and complete with respect to $H\Box$ models (condensed $H\Box$ models, strictly condensed $H\Box$ models).

The language $L\Diamond$ differs from $L\Box$ only in having $\Diamond$ instead of $\Box$, and the system $HK\Diamond$ is an extension of the Heyting propositional calculus in $L\Diamond$ with

$$\begin{array}{ll} \Diamond\,1.\ \Diamond(A \vee B) \to \Diamond A \vee \Diamond B & \\ & R\Diamond.\ \dfrac{A \to B}{\Diamond A \to \Diamond B}. \\ \Diamond\,2.\ \neg\Diamond\neg(A \to A) & \end{array}$$

A $H\Diamond$ frame (model) differs from a $H\Box$ frame (model) only in having $R_I^{-1}R_M \subseteq R_M R_I^{-1}$ instead of $R_I R_M \subseteq R_M R_I$. The relation $\models$ involves R_M for $\Diamond$ and is otherwise unchanged. A $H\Diamond$ frame (model) is condensed iff $R_I^{-1}R_M = R_M$, and it is strictly condensed iff $R_I^{-1}R_M = R_M R_I^{-1} = R_M$.

The system $HK\Diamond$ is sound and complete with respect to $H\Diamond$ models (condensed $H\Diamond$ models, strictly condensed $H\Diamond$ models).

The System $HK\Diamond'$

2. The syntax of $HK\Diamond'$. Let the language $L\Diamond'$ be obtained from $L\Box$ by replacing $\Box$ by the modal operator $\Diamond'$, read intuitively as "it is impossible that". We introduce the propositional calculus $HK\Diamond'$ in $L\Diamond'$ by extending the axiomatization of the Heyting propositional calculus given in [**1**] (see section 2; this axiomatization is given by $H1$–$H10$ and MP) with

$$\begin{array}{ll} \Diamond'\,1.\ \Diamond' A \wedge \Diamond' B \to \Diamond'(A \vee B) & \\ & R\Diamond'.\ \dfrac{A \to B}{\Diamond' B \to \Diamond' A}. \\ \Diamond'\,2.\ \Diamond'\neg(A \to A) & \end{array}$$

By a simple induction on the complexity of formulae it can be shown that any extension S of $HK\Diamond'$ (an extension being closed under the primitive rules of $HK\Diamond'$) is closed under the rule of replacement of equivalent formulae. It is also easy to show that the Deduction Theorem holds with respect to $\vdash_S$. We can also show that the theorems of $HK\Diamond'$ make a conservative, and hence consistent, extension of the Heyting propositional calculus in $L\Diamond'$ without $\Diamond'$ by a mapping e which replaces every $\Diamond'$ in a formula by $\neg$, and otherwise leaves the formula unchanged – for if $\vdash_{HK\Diamond'} A$, then $e(A)$ is provable in the Heyting propositional calculus.

A set of formulae Φ has the disjunction property iff $A \vee B \in \Phi$ implies $A \in \Phi$ or $B \in \Phi$. A system has this property iff the set of its theorems has this property. In order to prove that $HK\Diamond'$ has the disjunction property we can use the variant of Kleene's slash defined in [**1**] modified by taking $|\Diamond' A \Leftrightarrow_{df} |\neg A (\Leftrightarrow_{df}$ *not* $\Vdash A$, i.e. *not* $(\vdash A$ *and* $|A))$. The proof of the following lemma is then an immediate consequence of the fact that $\vdash_{HK\Diamond'} A$ implies that $e(A)$ is a theorem of the Heyting propositional calculus in $L\Diamond'$ without $\Diamond'$.

Lemma 1. *$HK\Diamond'$ has the disjunction property.*

3. $H\Diamond'$ models. A $H\Diamond'$ frame (model) is defined as a $H\Box$ frame (model) save that the condition $R_IR_M \subseteq R_MR_I$ is replaced by $R_IR_M \subseteq R_MR_I^{-1}$. In the definition of $\models$ we add the clause

$$x \models \Diamond' B \Leftrightarrow_{df} \forall y(xR_My \Rightarrow y \models B).$$

Note that the right-hand side of this clause coincides with the right-hand side of the clause for negation upon replacement of R_M by R_I. Next we show the following lemma.

Lemma 2. (Intuitionistic Heredity) *In every $H\Diamond'$ model $\langle X, R_I, R_M, V\rangle$, for every $x, y \in X$ and for every A of $L\Diamond'$, $xR_Iy \Rightarrow (x \models A \Rightarrow y \models A)$.*

Proof. By induction on the complexity of A. We shall consider only the case where A is of the form $\Diamond' B$. Suppose xR_Iy *and* $x \models \Diamond' B$. Next suppose that yR_Mz. With xR_My we obtain xR_IR_Mz, and using $R_IR_M \subseteq R_MR_I^{-1}$, it follows that $xR_MR_I^{-1}z$, i.e., for some t, xR_Mt *and* zR_It. Since $x \models \Diamond' B$ implies $\forall u(xR_Mu \Rightarrow u \not\models B)$ we have $t \not\models B$, which with zR_It and the induction hypothesis gives $z \not\models B$. Hence, $\forall z(yR_Mz \Rightarrow z \not\models B)$, which implies $y \models \Diamond' B$. q.e.d.

Lemma 2 shows in a certain sense the *sufficiency* of the condition $R_IR_M \subseteq R_MR_I^{-1}$ for making every $H\Diamond'$ model a model for the Heyting propositional calculus in the extended language $L\Diamond'$. That this condition is also *necessary* is shown by the following lemma.

Lemma 3. *Let $\langle X, R_I, R_M\rangle$ be as a $H\Diamond'$ frame save that not $R_IR_M \subseteq R_MR_I^{-1}$. Then there is a formula A of $L\Diamond'$ and a valuation V such that in $\langle X, R_I, R_M, V\rangle$ for some $x, y \in X$, xR_Iy and $x \models A$ and $y \not\models A$.*

Proof. Since *not* $R_IR_M \subseteq R_MR_I^{-1}$, there are some x, y and z such that

$$xR_Iy \textit{ and } yR_Mz \textit{ and } \forall t(xR_Mt \Rightarrow \textit{ not } zR_It).$$

Let $\forall u(u \models p \Leftrightarrow zR_Iu)$. It can easily be checked that there is a valuation such that this is satisfied (cf. Lemma 4(i) of [**3**]. From the last conjunct of (1) it follows that with this valuation $x \models \Diamond' p$. On the other hand, since zR_Iz, we have $y \not\models \Diamond' p$. q.e.d.

In a certain sense we have shown that models with the condition $R_IR_M \subseteq R_MR_I^{-1}$ form the largest class of models with respect to which we can expect to show that $HK\Diamond'$ is sound and complete. But we also have the following lemmata which indicate that a proper subclass of $N\Diamond'$ models might be sufficient.

Lemma 4. *In $H\Diamond'$ models, $x \models \Diamond' A \Leftrightarrow \forall y(xR_MR_I^{-1}y \Rightarrow y \not\models A)$.*

This lemma is easily proved by using $z \not\models A \Leftrightarrow y(yR_Iz \Rightarrow y \not\models A)$, which follows from Intuitionistic Heredity and the reflexivity of R_I. We can also show the following by using the reflexivity and transitivity of R_I.

LEMMA 5. *In the definition of $H\Diamond'$ frames we can replace the clause $R_I R_M \subseteq R_M R_I^{-1}$ by $R_I R_M R_I^{-1} \subseteq R_M R_I^{-1}$ yielding the same class of frames.*

So, roughly speaking, out of $H\Diamond'$ models we can make new models by replacing the $R_M R_I^{-1}$ relation by a new relation $R\Diamond'$ such that in these new models $x \models \Diamond' A \Leftrightarrow \forall y(xR\Diamond' y \Rightarrow y \not\models A)$ and $R_I R\Diamond' \subseteq R\Diamond'$. The relation $R\Diamond'$ is the R_M relation of the new models, which validate exactly the same formulae as the old ones. Since $R_M R_I^{-1} R_I^{-1} \subseteq R_M R_I^{-1}$, we can further "condense" these models by making $R\Diamond' R_I^{-1} \subseteq R\Diamond'$.

4. Soundness and completes of $HK\Diamond'$. In this section we shall prove that $HK\Diamond'$ is sound and complete with respect to $H\Diamond'$ models, and also with respect to some specific subclasses of $H\Diamond'$ models.

We say that a set of formulae Γ is nice iff Γ is consistent, deductively closed (i.e. $Cl(\Gamma) \subseteq \Gamma$) and it has the disjunction property. We build the canonical S frame $\langle X^c, R_I^c, R_M^c \rangle$ (canonical S model $\langle X^c, R_I^c, R_M^c, V^c \rangle$) by taking for X^c the set of all sets which are nice with respect to the system S, whereas $\Gamma R_I^c \Delta$ is defined as $\Gamma \subseteq \Delta$ ($V^c(p)$ is defined as $\{\Gamma \in X^c | p \in \Gamma\}$). It remains only to define R_M^c.

When S was a system in $L\Box$ we had in [**1**]

$$\Gamma R_M^c \Delta \Leftrightarrow_{df} \forall A(A \in \Delta \Rightarrow \Diamond A \in \Gamma).$$

The implication from left to right in this equivalence amounts to $\Box A \in \Gamma \Rightarrow (\Gamma R_M^c \Delta \Rightarrow A \in \Delta)$, which must be satisfied if we want $\Gamma \models B \Leftrightarrow B \in \Gamma$ to hold for the canonical S model. Similarly, when S was a system in $L\Diamond$ we had in [**1**]

$$\Gamma R_M^c \Delta \Leftrightarrow_{df} \forall A(A \in \Delta \Rightarrow \Diamond A \in \Gamma).$$

The implication from left to right in this equivalence amounts to $\Gamma R_M^c \Delta$ *and* $A \in \Delta \Rightarrow \Diamond A \in \Gamma$, which must be satisfied if we want $\Gamma \models B \Leftrightarrow B \in \Gamma$ to hold for the canonical S model. Now, when we have a system S in $L\Diamond'$, to prove $\Gamma \models B \Leftrightarrow B \in \Gamma$ for the canonical S model we must have $\Diamond' A \in \Gamma \Rightarrow (\Gamma R_M^c \Delta \Rightarrow A \notin \Delta)$. So, we shall use the following definition

$$\Gamma R_M^c \Delta \Leftrightarrow_{df} \forall A(\Diamond' A \in \Gamma \Rightarrow A \notin \Delta).$$

If $\Gamma\Diamond'$ is short for $\{A | \Diamond' A \in \Gamma\}$, then the right-hand side of this definition amounts to $\Gamma\Diamond' \cap \Delta = \emptyset$.

In the following two lemmata S shall stand for any *consistent* extension of $HK\Diamond'$ in $L\Diamond'$.

LEMMA 6. *The canonical S frame (model) is a $H\Diamond'$ frame (model).*

Proof. We have that $X^c \neq \emptyset$ since the set of theorems of S is consistent, and hence by Lemma 6 of [**1**], which holds in this context too, it can be extended to a nice set. (The set of theorems of $HK\Diamond'$ is already nice, since it is consistent, deductively closed and by Lemma 1 it has the disjunction property.) Since it is obvious that R^c_I is reflexive and transitive, it remains only to show the following

$$\begin{aligned}\Gamma R^c_I R^c_M \Delta &\Rightarrow \exists\Theta(\Gamma \subseteq \Theta \ \textit{and} \ \Theta\lozenge' \cap \Delta = \emptyset)\\ &\Rightarrow \lozenge' \cap \Delta = \emptyset\\ &\Rightarrow \Gamma R^c_M \Delta\\ &\Rightarrow \Gamma R^c_M (R^c_I)^{-1}\Delta, \ \text{by the reflexivity of } R^c_I.\end{aligned}$$

So the canonical S frame is a $H\lozenge'$ frame. For the rest see Lemma 7 of [**1**]. q.e.d.

Note that in the proof of Lemma 6 we have shown that the canonical S frame is a condensed $H\Box$ frame.

LEMMA 7. *In the canonical S model, for every $\Gamma \in X^c$ and for every A, $\Gamma \models A \Leftrightarrow A \in \Gamma$.*

Proof. By induction on the complexity of A. The basis, when A is p is trivial. In the induction step we shall consider only the modal case. We have

$$\begin{aligned}\Gamma \models \lozenge' B &\Leftrightarrow \forall\Delta(\Gamma R^c_M \Delta \Rightarrow \Delta \not\models B)\\ &\Leftrightarrow \forall\Delta(\Gamma\lozenge' \cap \Delta = \emptyset \Rightarrow B \notin \Delta), \ \text{using the induction hypothesis.}\end{aligned}$$

We shall show that $\lozenge' B \in \Gamma \Leftrightarrow \forall\Delta(\Gamma\lozenge' \cap \Delta = \emptyset \Rightarrow B \notin \Delta)$. From left to right this is obvious. For the other direction suppose $\lozenge' B \notin \Gamma$. Then we show that there is a nice Δ such that $\Gamma\lozenge' \cap \Delta = \emptyset$ *and* $B \in \Delta$.

Let $Z = \{\Phi | \Gamma\lozenge' \cap \Phi = \emptyset \ \textit{and} \ B \in \Phi \ \textit{and} \ Cl(\Phi) \subseteq \Phi\}$. First we show that $Cl(\{B\}) \in Z$. The only difficult part of this is to show that $\Gamma\lozenge' \cap Cl(\{B\}) = \emptyset$. Suppose $C \in \Gamma\lozenge'$ *and* $C \in Cl(\{B\})$. Then $\lozenge' C \in \Gamma$ *and* $\{B\} \vdash C$, from which we obtain $\vdash \lozenge' C \to \lozenge' B$ using the Deduction Theorem and $R\lozenge'$. But then, since Γ is nice, $\lozenge' B \in \Gamma$, and this is a contradiction. Hence, Z is nonempty, and it is easy to show that it is closed under unions of nonempty chains. So by Zorn's Lemma Z has a maximal element Δ with respect to $\subseteq$. We show first that

(i) Δ is consistent.

Otherwise, $\Delta \vdash \neg(C \to C)$, and since $Cl(\Delta) \subseteq \Delta$, we have $\neg(C \to C) \in \Delta$. But since Γ is nice $\lozenge'\neg(C \to C) \in \Gamma$, which contradicts $\Gamma\lozenge' \cap \Delta = \emptyset$ ($\lozenge'\neg(C \to C)$ is $\lozenge' 2$). Next we infer immediately from $\Delta \in Z$ that

(ii) Δ is deductively closed.

Now suppose that Δ doesn't have the disjunction property, i.e., for some C and D, $C \vee D \in D$ *and* $C \notin \Delta$ *and* $D \notin \Delta$. Since $\Delta \cup \{C\}$ and $\Delta \cup \{D\}$ are proper supersets of Δ, they cannot be in Z. *A fortiori*, $Cl(\Delta \cup \{C\})$ and $Cl(\Delta \cup \{D\})$ are not in Z. This is possible only if for some C_1 from the first and some D_1 from the second of these last two sets, $\lozenge' C_1 \in \Gamma$ *and* $\lozenge' D_1 \in \Gamma$. Since Γ is nice, using $\lozenge' 1$ we obtain $\lozenge'(C_1 \vee D_1) \in \Gamma$. On the other hand, we have

$$\begin{aligned}\Delta \cup \{C\} \vdash C_1 \ \textit{and} \ \Delta \cup \{D\} \vdash D_1 &\Rightarrow \Delta \vdash C \vee D \to C_1 \vee D_1\\ &\Rightarrow \Delta \vdash C_1 \vee D_1, \ \text{since } C \vee D \in \Delta\\ &\Rightarrow C_1 \vee D_1 \in \Delta, \ \text{by (ii)}\end{aligned}$$

which contradicts $\Gamma\Diamond' \cap \Delta = \emptyset$. So

(iii) Δ has the disjunction property

and we can conclude that Δ is nice. q.e.d.

Now we can prove the soundness and completeness of $HK\Diamond'$ with respect to $H\Diamond'$ models.

THEOREM 1. $\vdash_{HK\Diamond'} A \Leftrightarrow$ *for every* $H\Diamond'$ *frame* Fr, $Fr \vdash A$.

The soundness part ($\Rightarrow$) of this theorem is proved by a straightforward induction on the length of proof of A in $HK\Diamond'$. For the completeness part ($\Leftarrow$) we proceed analogously to the proof of the ($\Leftarrow$) part of Theorem 1 of [**1**].

Next we shall consider narrower classes of models with respect to which $HK\Diamond'$ can be proved sound and complete. We alluded to the possibility of such classes at the end of section 3. We can prove the following theorem.

THEOREM 2. $\vdash_{HK\Diamond'} A \Leftrightarrow$ *for every condensed* $H\Box$ *frame* Fr, $Fr \models A$.
$\Leftrightarrow$ *for every condensed* $H\Box$ *frame* Fr *in which* $R_M R_I^{-1} = R_M$, $Fr \models A$.

Proof. From left to right we use the ($\Rightarrow$) part of Theorem 1. For the other direction it is enough to establish the following. The canonical $HK\Diamond'$ frame is a condensed $H\Box$ frame, as we have remarked after the proof of Lemma 6. To show that $R^c_M(R^c_I)^{-1} \subseteq R^c_M$ we have

$$\begin{aligned}\Gamma R^c_M(R^c_I)\Delta &\Rightarrow \exists\Theta(\Gamma\Diamond' \cap \Theta = \emptyset \ \textit{and} \ \Delta \subseteq \Theta \\ &\Rightarrow \Gamma\Diamond' \cap \Delta = \emptyset \\ &\Rightarrow \Gamma R^c_M \Delta.\end{aligned}$$

The converse is trivial. q.e.d.

It is easily shown that in condensed $H\Box$ frames $\Box\neg A \rightarrow \Diamond' A$ is valid, but $\Diamond' A \rightarrow \Box\neg A$ is not.

The System $HK\Box'$

5. The syntax of $HK\Box'$. Let the language $L\Box'$ be obtained from $L\Box$ by replacing $\Box$ by the modal operator $\Box'$, read intuitively as "it is not necessary that". We introduce the propositional calculus $HK\Box'$ by extending the Heyting propositional calculus in $L\Box'$ with

$\Box'$ 1. $\Box'(A \wedge B) \rightarrow \Box' A \vee \Box' B$

$\Box'$ 2. $\neg\Box'(A \rightarrow A)$

$$R\Box'. \ \frac{A \rightarrow B}{\Box' B \rightarrow \Box' A}.$$

As before, any extension S of $HK\Box'$ is closed under the rule of replacement of equivalent formulae and the Deduction Theorem holds with respect to $\vdash_S$. The mapping e (see section 2) can show that $HK\Box'$ is consistent — for if $\vdash_{HK\Box'} A$, then $e(A)$ is provable in the classical propositional calculus; but this mapping cannot show as before that $HK\Box'$ is a conservative extension of the Heyting propositional calculus in $L\Box'$ without $\Box'$. To show this conservativeness we shall use a model-theoretical argument (see section 7).

Before proving that $HK\Box'$ has the disjunction property we first establish the following lemma.

LEMMA 8. *No formula of the form $\Box' A$ is a theorem of $HK\Box'$.*

Proof. Consider the mapping f which replaces subformulae of the form $\Box' A$ by $\neg(A \to A)$ (cf. Lemma 14 of [**1**]). It is easy to show that $\vdash_{HK\Box'} A$ implies that $f(A)$ is a theorem of the Heyting propositional calculus, from which the Lemma follows. q.e.d.

Next we use this lemma and the variant of Kleene's slash defined in [**1**] modified by taking $|\Box' A \Leftrightarrow_{df} |\neg A$ (we could as well have taken $|\Box' A \Leftrightarrow_{df} B$, where B is arbitrary) in order to prove that $\vdash_{HK\Box'} A \Rightarrow |A$, which establishes the following lemma.

LEMMA 9. *$HK\Box'$ has the disjunction property.*

6. $H\Box'$ **models.** A $H\Box'$ frame (model) is defined as a $H\Box$ frame (model) save that the condition $R_I R_M \subseteq R_M R_I$ is replaced by $R_I^{-1} R_M \subseteq R_M R_I$. In the definition of $\models$ we add the clause

$$x \models \Box' B \Leftrightarrow_{df} \exists y(xR_M y \text{ and } y \not\models B).$$

Next, analogously to Lemmata 2 and 3 we can prove the following lemmata which establish that the condition $R_I^{-1} R_M \subseteq R_M R_I$ is necessary and sufficient to make every $H\Box'$ model a model for the Heyting propositional calculus in $L\Box'$.

LEMMA 10 (Intuitionistic Heredity). *In every $H\Box'$ model $\langle X, R_I R_M, V\rangle$, for every $x, y \in X$ and for every A of $L\Box'$, $xR_I y \Rightarrow (x \models A \Rightarrow y \models A)$.*

LEMMA 11. *Let $\langle X, R_I, R_M\rangle$ be as a $H\Box'$ frame save that not $R_I^{-1} R_M \subseteq R_M R_I$. Then there is a formula A of $L\Box'$ and a valuation V such that in $\langle X, R_I, R_M, V\rangle$ for some $x, y \in X$, $xR_I y$ and $x \models A$ and $y \not\models A$.*

In a certain sense we have shown that models with the condition $R_I^{-1} R_M \subseteq R_M R_I$ form the largest class of models with respect to which we can expect to show that $HK\Box'$ is sound and complete. But we also have the following lemmata which indicate that a proper subclass of $H\Box'$ models might be sufficient.

LEMMA 12. *In $H\Box'$ models, $x \models \Box' A \Leftrightarrow \exists y(xR_M R_I y$ and $y \not\models A)$.*

This lemma is easily proved using $z \not\models A \Leftrightarrow \exists y(zR_I y$ and $y \not\models A)$, which follows from Intuitionistic Heredity and the reflexivity of R_I. We can also easily show the following using the reflexivity and transitivity of R_I.

LEMMA 13. *In the definition of $H\Box'$ frames we can replace the clause $R_I^{-1} R_M \subseteq R_M R_I$ by $R_I^{-1} R_M R_I \subseteq R_M R_I$ yielding the same class of frames.*

So, roughly speaking, out of $H\Box'$ models we can make new models by replacing the $R_M R_I$ relation by a new relation $R\Box'$ such that in these new models $x \models \Box' A \Leftrightarrow \exists y(xR\Box' y \textit{ and } y \not\models A)$ and $R_I^{-1} R\Box' \subseteq R\Box'$. The relation $R\Box'$ is the R_M relation of the new models, which validate exactly the same formulae as the old ones. Since $R_M R_I R_I \subseteq R_M R_I$ we can further "condense", these new models by making $R\Box' R_I \subseteq R\Box'$.

7. Soundness and completeness of $HK\Box'$. In this section we shall prove that $HK\Box'$ is sound and complete with respect to $H\Box'$ models, and also with respect to some specific subclasses of $H\Box'$ models.

For a system S in $L\Box'$ to prove $\Gamma \models B \Leftrightarrow B \in \Gamma$ for the canonical S model we must have $\Gamma R^c_M \Delta$ *and* $A \notin \Delta \Rightarrow \Box' A \in \Gamma$. So, we shall define the canonical S frame (model) as before save that for R^c_M we have

$$\Gamma R^c_M \Delta \Leftrightarrow_{df} \forall A(A \notin \Delta \Rightarrow \Box' A \in \Gamma).$$

If $\Gamma\Box'$ is short for $\{A | \Box' A \in \Gamma\}$ and 1 is the set of all formulae of $L\Box'$, then the right-hand side of this definition amounts to $\Gamma\Box' \cup \Delta = 1$.

The following two lemmata about consistent extensions S of $HK\Box'$ in $L\Box'$ are proved analogously to Lemmata 6 and 7.

LEMMA 14. *The canonical S frame (model) is a $H\Box'$ frame (model).*

To prove this lemma we have

$$\begin{aligned} \Gamma(R^c_I)^{-1} R^c_M \Delta &\Rightarrow \exists\Theta(\Theta \subseteq \Gamma \textit{ and } \Gamma\Box' \cup \Delta = 1) \\ &\Rightarrow \Gamma\Box' \cup \Delta = 1 \\ &\Rightarrow \Gamma R^c_M \Delta \\ &\Rightarrow \Gamma R^c_M R^c_I \Delta \end{aligned}$$

which shows also that the canonical S frame is a condensed $H\Diamond$ frame.

LEMMA 15. *In the canonical S model, for every $\Gamma \in X^c$ and for every $A, \Gamma \models A \Leftrightarrow A \in \Gamma$.*

Proof. By induction on the complexity of A. The basis, when A is p, is trivial. For the modal case of the induction step we have

$$\begin{aligned} \Gamma \models B &\Leftrightarrow \exists\Delta(\Gamma R^c_M \Delta \textit{ and } \Delta \not\models B) \\ &\Leftrightarrow \exists\Delta(\Gamma\Box' \cup D = 1 \textit{ and } B \notin \Delta), \text{ using the induction hypothesis.} \end{aligned}$$

We shall show that $\Box' B \in \Gamma \Leftrightarrow \exists\Delta(\Gamma\Box' \cup \Delta = 1 \textit{ and } B \notin \Delta)$. From right to left this is obvious.

For the other direction suppose $\Box' B \in \Gamma$. Let Θ be the complement of $\Gamma\Box'$. This set is nonempty since, Γ being nice, by $\Box' 2$, $\neg\Box'(C \to C) \in \Gamma$ – hence,

$\Box'(C \to C) \notin \Gamma$, and hence $C \to C \in \Theta$. We show that *not* $\Theta \vdash B$. Otherwise for some $B_1, \dots, B_n \in \Theta$, $n \geq 1$ (here we use the nonemptiness of Θ)

$$\begin{array}{ll} \vdash B_1 \wedge \cdots \wedge B_n \to B & \\ \vdash \Box' B \to \Box'(B_1 \wedge \cdots \wedge B_n) & \text{, by using } R\Box' \\ \Gamma \vdash \Box'(B_1 \wedge \cdots \wedge B_n) & \text{, since } \Box' B \in \Gamma \\ \Gamma \vdash \Box' B_1 \vee \cdots \vee \Box' B_n & \text{, by using } \Box' 1 \\ \exists i.\ \Box' B_i \in \Gamma & \text{, since } \Gamma \text{ is nice.} \end{array}$$

But then $B_i \in \Gamma\Box'$, which contradicts the assumption $B_i \in \Theta$. Since *not* $\Theta \vdash B$, by using Lemma 6 of [**1**], which holds in this context too, Θ can be extended to a nice set Δ such that $B \notin \Delta$. q.e.d.

Now we can prove the soundness and completeness of $NK\Box'$ with respect to $H\Box'$ models.

THEOREM 3. $\vdash_{HK\Box'} A \Leftrightarrow$ *for every $H\Box'$ frame Fr, $Fr \models A$.*

The proof of this theorem is analogous to the proof of Theorem 1.

Having established this theorem, we can easily show that $HK\Box'$ is a conservative extension of the Heyting propositional calculus in $L\Box'$ without $\Box'$. A Kripke model falsifying a non-theorem of this calculus is a $H\Box'$ model where R_M is empty, modulo some inessential adjustments.

Next we shall consider narrower classes of models with respect to which $HK\Box'$ can be proved sound and complete. We alluded to the possibility of such classes at the end of section 6. We can prove the following theorem.

THEOREM 4. $\vdash_{HK\Box'} A \Leftrightarrow$ *for every condensed $H\Diamond$ frame Fr, $Fr \models A$*
$\Leftrightarrow$ *for every condensed $H\Diamond$ frame Fr in which*
$R_M R_I = R_M$, $Fr \models A$.

Proof. From left to right we use the ($\Rightarrow$) part of Theorem 3. For the other direction it is enough to establish the following. The canonical $HK\Box'$ frame is a condensed $H\Diamond$ frame, as we have remarked after Lemma 14. To show that $R^c_M R^c_I R \subseteq R_M$ we have

$$\begin{aligned} \Gamma R^c_M R^c_I \Delta &\Rightarrow \exists\Theta(\Gamma\Box' \cup \Theta = 1 \ \textit{and} \ \Theta \subseteq \Delta) \\ &\Rightarrow \Gamma\Box' \cup \Delta = 1 \\ &\Rightarrow \Gamma R^c_M \Delta. \end{aligned}$$

The converse is trivial. q.e.d.

It is easily shown that in condensed $H\Diamond$ frames $\Diamond\neg A \to \Box' A$ is valid, but $\Box' A \to \Diamond\neg A$ is not.

Although $HK\Box'$ has the disjunction property, the intuitively quite plausible extension of $HK\Box'$ with $\Box'(A \wedge \neg A)$ will not have it. In this extension using

$\Box'1$ we obtain $\Box' A \vee \Box' \neg A$. It is easy to show that for a $H\Box'$ frame Fr, $Fr \models \Box'(A \vee \neg A) \Leftrightarrow \forall x \exists y . x R_M y$, i.e., $\Box'(A \wedge \neg A)$ (or $\Box' \neg(A \rightarrow A)$) is equivalent to the seriality of R_M in $H\Box'$ frames. In [**3**] we have seen that both $HK\Box$ and $HK\Diamond$ can be extended with axioms equivalent to the seriality of R_M in appropriate frames without spoiling the disjunction property. The same holds for $HK\Diamond'$: the schema with $\Diamond'$ equivalent to the seriality of R_M in $H\Diamond'$ frames is $\neg\Diamond'(A \rightarrow A)$,and $HK\Diamond'$ can be extended with it without spoiling the disjunction property.

Intuitionistic Negation as a Modal Operator

8. The system $Hn\Diamond'$. In this section we shall present a system containing $HK\Diamond'$ where $\Diamond'$ A is equivalent to $\neg A$, and $\neg$ is intuitionistic negation. Let $Hn\Diamond'$ ("n' stands for "negation") be the system in $L\Diamond'$ obtained by extending the Heyting propositional calculus with

$$\Diamond' n. \ \Diamond' A \leftrightarrow \neg A.$$

It is easy to see that alternatively $Hn\Diamond'$ could be axiomatized by taking the schemata $\Diamond' A \rightarrow (A \rightarrow B)$ and $(A \rightarrow \Diamond' A) \rightarrow \Diamond' A$ instead of $\Diamond' n$. It is also easy to see that $HK\Diamond'$ is a proper subsystem of $Hn\Diamond'$. We can show the following lemma.

LEMMA 16. *Let Fr be a $H\Diamond'$ frame. Then*

$$Fr \models \Diamond' A \rightarrow \neg A \Leftrightarrow R_I R_I^{-1} \subseteq R_M R_I^{-1}, \ Fr \models \neg A \rightarrow \Diamond' A \Leftrightarrow R_M R_I^{-1} \subseteq R_I R_I^{-1}.$$

Proof. For the first ($\Rightarrow$) part suppose that for some x and y, $x R_I R_I^{-1} y$ *and not* $x R_M R_I^{-1} y$. Then take a valuation such that $\forall u (u \models p \Leftrightarrow$ *not* $x R_M R_I^{-1} u)$ (cf. Lemma 4 (iii) of [**3**]). With this valuation $x \not\models \Diamond' p \rightarrow \neg p$. For the second ($\Rightarrow$) part we proceed analogously, and it is easy to check the ($\Leftarrow$) parts. q.e.d.

It can easily be shown that if *Fr* is a $H\Diamond'$ frame

$$Fr \models \Diamond' A \rightarrow A \Leftrightarrow Fr \models \Diamond' A \rightarrow (A \rightarrow B)$$
$$\Leftrightarrow R_M R_I^{-1} \text{ is reflexive.}$$

Let $H\Diamond'$ frames (models) in which $R_I R_I^{-1} = R_M R_I^{-1}$ be called $Hn\Diamond'$ *frames (models).* In $Hn\Diamond'$ frames $R_M R_I^{-1}$ is reflexive, as we have just remarked, and it is also symmetric, since $R_I R_I^{-1}$ is symmetric. It can easily be shown that if *Fr* is a $H\Diamond'$ frame

$$Fr \models A \rightarrow \Diamond' \Diamond' A \Leftrightarrow R_M R_I^{-1} \text{ is symmetric.}$$

Next we give the following lemma.

LEMMA 17. *The canonical $Hn\Diamond'$ frame (model) is a $Hn\Diamond'$ frame (model).*

Since we have Lemma 6, it is enough to establish that in the canonical $Hn\Diamond'$ frame $R_I^c (R_I^c)^{-1} = R_M^c (R_I^c)^{-1}$. For that we use methods applice in the proof of Lemma 7, and also Lemma 6 of [**1**], which holds in the present context too. With the help of Lemma 17, by procedures applied before we can establish the following theorem.

THEOREM 5. $\vdash_{Hn\Diamond'} A \Leftrightarrow$ *for every* $Hn\Diamond'$ *frame* Fr, $Fr \models A$
$\Leftrightarrow$ *for every condensed* $H\Box$ *frame* Fr *in which* $R_I R_I^{-1} = R_M R_I^{-1}$, $Fr \models A$
$\Leftrightarrow$ *for every* $Fr = \langle X, R_I, R_M \rangle$ *in which* $X \neq \emptyset$, R_I *is reflexive and transitive and* $R_I R_I^{-1} = R_M$, $Fr \models A$.

For the last equivalence we use the fact that $R_I R_I^{-1} = R_M$ implies $R_I R_M = R_M$ and $R_M R_I^{-1} = R_M$, which follows from the reflexivity and transitivity of R_I. This last equivalence is probably the most interesting. It shows that if we take an ordinary Kripke model for the Heyting propositional calculus $\langle X, R_I, V \rangle$ where R_I is reflexive and transitive, and take every $R_I R_I^{-1}$ relation to be an R_M relation, we obtain a model in which impossibility amounts to intuitionistic negation. (This model is also a condensed $H\Box$ model.) This is connected with the fact that in ordinary Kripke models $x \models \neg A \Leftrightarrow \forall y (x R_I R_I^{-1} y \Rightarrow y \not\models A)$, which is easily shown with the help of Intuitionistic Heredity and the reflexivity of R_I. It is immediately seen that the R_M relation so defined is reflexive and symmetric, but not necessarily transitive, which points towards a certain connection between intuitionistic negation and the *Brouwersche* modal logic B (based on the classical propositional calculus), for which Kripke frames $\langle X, R_M \rangle$ where R_M is reflexive and symmetric are characteristic. Historically, B was connected with intuitionistic negation because $A \rightarrow \neg\Diamond\neg\Diamond A$ is provable in B, but the converse is not (see [**5**], p. 58, fn. 37). This comment on the third equivalence of Theorem 5 doesn't depend essentially on assuming for the frames $\langle X, R_I \rangle$ of ordinary Kripke models for the Heyting propositional calculus that R_I is only reflexive and transitive: we could as well have used narrower classes of frames — like, for example, trees — with the same effect.

It is, of course, trivial to show that $Hn\Diamond'$ is also sound and complete with respect to models with frames $\langle X, R_I, R_M \rangle$ where $X \neq \emptyset$, R_I is reflexive and transitive and $R_I = R_M$. These frames are also condensed $H\Box$ frames, but in them $R_M R_I^{-1} \not\subseteq R_M$. If in $H\Diamond'$ frames $R_I = R_M$, then $\Diamond'$ collapses into intuitionistic negation. On the other hand, if in $H\Diamond'$ frames $R_I = R_M$ then $x \models \Box' A \Leftrightarrow x \not\models A$ and $R_I^{-1} R_I \subseteq R_I$ from which it follows that $\Box'$ collapses into classical negation, R_I is an equivalence relation, and everything reduces to classical logic.

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Matematički institut
Knez Mihailova 35
Beograd

(Received 26 04 1983)