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ON GRAPHS WHOSE SPECTRAL SPREAD DOES NOT EXCEED 4

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Abstract. In this paper all minimal graphs with the property of having the spectral spread greater than 4 are determined. In addition, all connected graphs whose spactral does not exceed 4 are described.

1. Introduction

All considered graphs are undirected graphs without loops or multiple edges. By eigenvalues of a graph G we mean eigenvalues of its 0–1 adjacency matrix A. The spectral spread (briefly the spread) $s(G)$ of G is the spread $s(A)$ of its adjacency matrix A, i.e. $s(G) = s(A) = r(G) - \lambda(G)$, where $r(G)$ and $\lambda(G)$ are the largest and the least eigenvalue of G. For definition and properties of the spread of matrices, one can consult [**3**].

J. H. Smith [**4**] has determined all graphs whose largest eigenvalue does not exceed 2; after that D. Cvetković, M. Doob and I. Gutman [**1**] have determined all minimal graphs with the property of having the largest eigenvalue greater than 2. In this paper we extend their results in some sense.

Let H be a proper induced subgraph of G. By virtue of the well known Interlacing Theorem (see [**2**] for example) it follows that $r(G) \geq r(H)$ and $\lambda(G) \leq \lambda(H)$, i.e. $s(G) \geq s(H)$. Thus for any real number $L > 0$ we may consider the graphs with $s(G) > L$ that are minimal with respect to that property. In this paper we shall find all such graphs for $L = 4$.

We consider also the following question: For a real number $L > 0$ find all graphs with $s(G) \leq L$. We give an explicit description of the connected graphs G satisfying $s(G) \leq 4$. Combining this with the results of Smith [**4**], we see that there are exactly five graphs with $r(G) > 2$ and $s(G) \leq 4$.

2. Minimal connected graphs with $s(G) > 4$

Recall that a graph is minimal with respect to (briefly w.r.t.) the property P if it has the property P and none of its proper induced subgraphs has this property.

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Now we determine all connected minimal graphs w.r.t. the property of having the spread greater than 4.

The determination of minimal trees with the spread greater than 4 is equivalent to the determination of minimal trees with the largest eigenvalue greater than 2.

LEMMA 1. (Cvetković, Doob, Gutman [**1**]) *There are exactly 9 minimal trees w.r.t. the property of having the spread greater than* 4 *and these are the graphs* $G_{22} - G_{30}$ *displayed in Fig.* 1.□

THEOREM 1. *There are exactly* 30 *minimal connected graphs w.r.t. the property of having the spread greater than* 4 *and they are displayed in Fig.* 1.

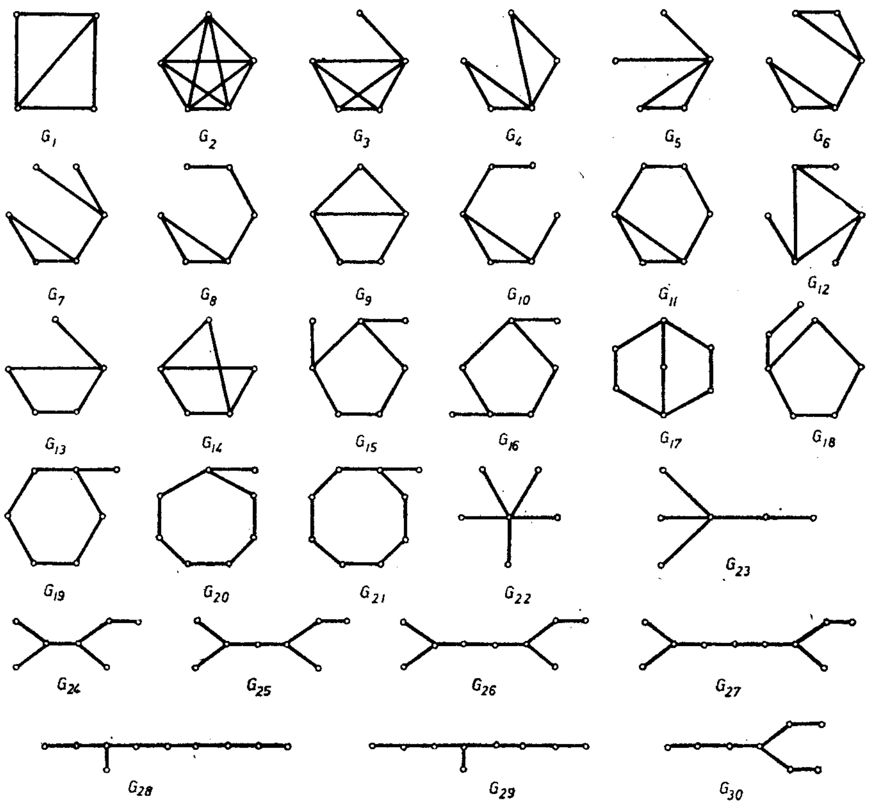

Fig. 1

Proof. It is easy to check that the graphs $G_1 - G_{30}$ in Fig. 1 are minimal graphs w.r.t. the property of having the spread greater than 4.

We prove that if a connected graph G is a minimal graph with the spread greater than 4, then G is one of the graphs $G_1 - G_{30}$ in Fig. 1. Lemma 1 takes care of the case when G is a tree. Hence, it is enough to consider the graphs with circuits.

Let G be a connected minimal graph with the spread greater than 4 and let n be the length of the shortest circuits of G. We denote the vertices of an arbitrary circuit C_n of G by $v_1, \dots, v_n$, so that the vertices v_i and $v_{i+1} (i = 1, \dots, n-1))$, v_1 and v_n are adjacent. Let $T_{i_1 \dots i_k} (1 \leq i_1 < \dots < i_k \leq n;\ 1 \leq k \leq n)$ be the set of vertices from $V(G) \backslash V(C_n)$ which are adjacent exactly to the vertices $v_{i_1}, \dots, v_{i_k}$ of C_n. Let T_0 be the set of vertices from $V(G) \backslash V(C_n)$ which are not adjacent to any vertex of C_n. We distinguish the following five cases:

Case 1: $n = 3$. If at least one of the sets $T_{ij} (1 \leq i < j \leq 3)$ is nonempty, then the graph G_1 is an induced subgraph, and hence eqauls G. Let $T_{ij} = \varnothing (1 \leq i < j \leq 3)$. Then if $\mid T_{123} \mid \geq 2$ and no two vertices of T_{123} are adjacent, G contains the proper induced subgraph G_1 contradicting the minimality condition. Thus, if $\mid T_{123} \mid \geq 2$, the graph G_2 is contained in (and hence equals) G. If $\mid T_{123} \mid = 1$, then G is G_3. Let $T_{123} = \varnothing$. If some of the sets T_i $(1 \leq i \leq 3)$ contains more than one vertex, then G is either G_4 or G_5. Let $\mid T_i \mid \leq 1$ $(1 \leq i \leq 3)$. Having in mind symmetric cases, we distinguish the following cases:

1° $\mid T_1 \mid = 1$, $T_2 = T_3 = \varnothing$. If the set T_0 contains two vertices adjacent to the vertex $x \in T_1$, then G is either G_6 or G_7. If T_0 contains only one vertex adjacent to the vertex $x \in T_1$, then G is G_8.

2° $\mid T_1 \mid = \mid T_2 \mid = 1$, $T_3 = \varnothing$. If the vertices $x \in T_1$ and $y \in T_2$ are adjacent, then G is G_9. If the vertices x and y are not adjacent, then G is either G_{10} or G_{11}.

3° $\mid T_1 \mid = \mid T_2 \mid = \mid T_3 \mid = 1$. If at least two vertices between $x \in T_1$, $y \in T_2$ and $z \in T_3$ are adjacent, then G contains the proper induced subgraph G_9 contradicting the minimality condition. Therefore G is G_{12}.

Case 2: $n = 4$. Then $T_{12} = T_{14} = T_{23} = T_{34} = T_{123} = T_{124} = T_{134} = T_{234} = T_{1234} = \varnothing$ because G has not triangles. If at least one of the sets $T_i (1 \leq i \leq 4)$ is nonempty, then G is G_{13}. Let $T_i = \varnothing (1 \leq i \leq 4)$. If $T_{13} \neq \varnothing$ and $T_{24} = \varnothing$, then G is G_{14}. If $T_{13} \neq \varnothing$ and $T_{24} \neq \varnothing$, then G contains the proper induced subgraph G_{14} contradicting the minimality condition.

In the orders cases $(n \geq 5)$ we have that $T_{i_1 \dots i_k} = \varnothing$ $(1 \leq i_1 < \dots < i_k \leq n;\ 2 \leq k \leq n)$, because G does not contain circuits whose length is less than n. If at least one of the sets $T_i (1 \leq i \leq n)$ contains more than one vertex, then G contains the proper induced subgraph G_{23} contradicting the minimality condition. Let $\mid T_i \mid \leq 1$ $(1 \leq i \leq n)$.

Case 3: $n = 5$. Now, if at least two of the sets $T_i (1 \leq i \leq 5)$ are nonempty, then G is one of the graphs G_{15}, G_{16}, G_{17}. If exactly one of the sets $T_i (1 \leq i \leq 5)$ is nonempty, then G is G_{18}.

Case 4: $6 \leq n \leq 8$. Then for $n = 6, 7, 8$ the graphs G_{19}, G_{20} and G_{21}, respectively, are contained in (and hence are equal to) G.

Case 5: $n \geq 9$. Then G contains the proper induced subgraph G_{29} contradicting the minimality condition.

This completes the proof of Theorem 1. □

3. Graphs whose spread does not exceed 4

In this section we determine all connected graphs whose spread does not exceed 4. In the proof of Theorem 2 we use the following lemma.

LEMMA 2. (Smith [**4**]) *Let G be a graph with the largest eigenvalue $r(G)$. Then $r(G) \leq 2$ if and only if each component of G is an induced subgraph of one of the graphs $H_6 - H_{11}$ displayed in Fig. 2.* □

THEOREM 2. *Let G be a connected graph with the spread $s(G)$. Then $s(G) \leq 4$ if and only if G is an induced subgraph of one of the graphs displayed in Fig.2.*

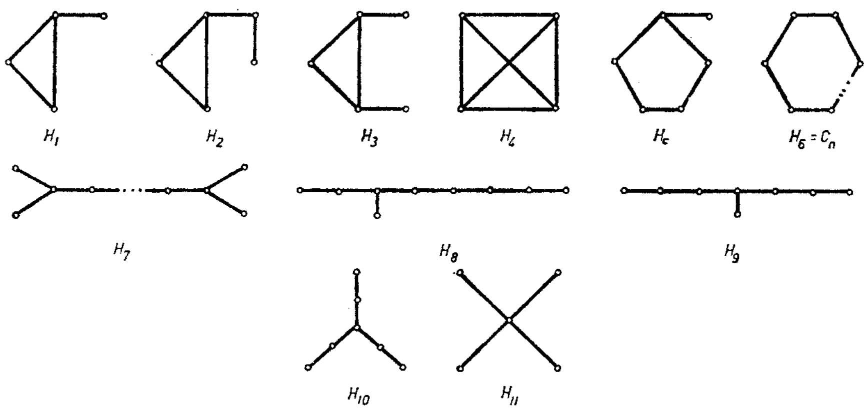

Fig. 2

Proof. Lemma 2 takes care of the case when G is a tree. Namely the determination of the trees with the spread less than or equal to 4 is equivalent to the determination of the trees whose largest eigenvalue does not exceed 2. Hence, it is enough to consider the graphs with circuits.

It is easy to check that the graphs $H_1 - H_{11}$ have the spread less than or equal to 4. Consequently, each induced subgraph of these graphs has the spread less than or equal 4.

Conversely, let G be a connected graph with circuits, whose spread does not exceed 4. To describe G, we use the method of impossible subgraphs. We note that G does not contain any of the graphs $G_1 - G_{30}$ depicted in Fig. 1, as an induced subgraph. The denotation and scheme of the proof are the same as in Theorem 1.

Let C_n be the smallest circuit in G. We distinguish the following four cases:

Case 1: $n = 3$. We have $T_{ij} = \varnothing\,(1 \le i < j \le 3)$ since otherwise G would contain the induced subgraph G_1. Moreover, the sets $T_i (1 \le i \le 3)$ can contain at the most one vertex (otherwise G would contain G_4 or G_5 as an induced subgraph); thus, $\mid T_i \mid \le 1 (1 \le i \le 3)$. Next, the set T_{123} can contain at the most one vertex, too (otherwise G would contain G_1 or G_2 as an induced subgraph).

By a direct verification, we determine the mutual relations between the corresponding sets. First, the vertices of T_i and $T_j (1 \le i < j \le 3)$ cannot be adjacent (otherwise G would contain G_9 as an induced subgraph). Next, the sets T_i and T_{123} are not consistent (otherwise G would contain G_1 or G_3 as an induced subgraph).

Taking into account all possible combinations and having in mind symmetry, we distinguish the following subcases:

1° $T_1 = T_2 = T_3 = T_{123} = \varnothing$. Then $G = C_3$.

2° $\mid T_1 \mid = 1,\ T_2 = T_3 = T_{123} = \varnothing$. Then $\mid T_0 \mid \le 1$, because otherwise G would contain at least one of the graphs G_6, G_7, G_8 as an induced subgraph. Thus, either $G = H_1$ or $G = H_2$.

3° $\mid T_{123} \mid = 1,\ T_1 = T_2 = T_3 = \varnothing$. In this case $T_0 = \varnothing$ (otherwise G would contain G_3 as an induced subgraph). Thus, $G = H_4$.

4° $\mid T_1 \mid = \mid T_2 \mid = 1,\ T_3 = T_{123} = \varnothing$. Then $T_0 = \varnothing$, because otherwise G would contain G_{10} or G_{11} as an induced subgraph. Hence, $G = H_3$.

We note that the combination $\mid T_1 \mid = \mid T_2 \mid = \mid T_3 \mid = 1,\ T_{123} = \varnothing$ is impossible. Indeed, in the contrary case G would contain G_{12} as an induced subgraph.

Case 2: $n = 4$. Then $T_i = \varnothing (1 \le i \le 4)$, because otherwise G would contain G_{13} as an induced subgraph. Besides, $T_{13} = T_{24} = \varnothing$, because in the contrary case G would contain G_{14} as an induced subgraph. Thus, $G = C_4$.

Case 3: $n = 5$. Then $\mid T_i \mid \le 1\ (1 \le i \le 5)$, because otherwise G would contain G_{23} as an induced subgraph. Moreover, the sets T_i and $T_j (1 \le i < j \le 5)$ are not consistent (in the contrary case G would contain at least one of the graphs G_{15}, G_{16}, G_{17} as an induced subgraph). Thus, we have only two possibilities:

1° $T_i = \varnothing (1 \le i \le 5)$. Then $G = C_5$.

2° $\mid T_1 \mid = 1,\ T_i = \varnothing (2 \le i \le 5)$. In this case $T_0 = \varnothing$, because otherwise G would contain G_{18} as an induced subgraph. Thus, $G = H_5$.

Case 4: $n \ge 6$. Then $T_i = \varnothing (1 \le i \le n)$. Indeed, in the contrary case G would contain the graphs G_{19}, G_{20} and G_{21} for $n = 6, 7, 8$, respectively, and the graph G_{29} for $n \ge 9$. Thus, $G = C_n$.

This completes the proof of the theorem. □

Corollary. *There are exactly five connected graphs with $r(G) > 2$ and $s(G) \le 4$ and these are the graphs $H_1 - H_5$ displayed in Fig. 2.*

4. Minimal disconnected graphs with $s(G) > 4$

In this section we determine all disconnected minimal graphs w.r.t. the property of having the spread greater than 4.

Let C_n be a circuit of length n, P_n a path of length $n-1$, S_n a star with $n+1$ vertices and $V_{m.n}$ and W_n the graphs displayed in Fig. 3.

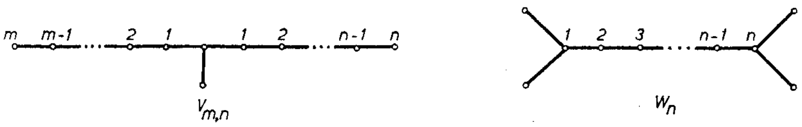

Fig. 3

Theorem 3. *There are exactly* 17 *disconnected minimal graphs w.r.t. the property of having the spread greater than* 4. *These are*

$$\begin{aligned}&H_1 \cup C_4,\ \ H_1 \cup C_6,\ \ H_1 \cup P_7,\ \ H_1 \cup V_{1,2},\ \ H_1 \cup S_4,\\ &H_2 \cup P_6,\ \ H_3 \cup P_5,\ \ H_3 \cup S_3,\ \ H_4 \cup P_3,\ \ H_5 \cup C_4,\\ &H_5 \cup C_6,\ \ H_5 \cup C_8,\ \ H_5 \cup P_9,\ \ H_5 \cup V_{1,3},\ \ H_5 \cup V_{2,2},\\ &H_5 \cup W_2,\ \ H_5 \cup S_4.\end{aligned} \tag{1}$$

Proof. It is easy to check that all graphs (1) are minimal graphs w.r.t. the property of having the spread greater than 4.

Let G be any disconnected minimal graphs with the spred greater than 4. Then G satisfies the following conditions:

1) G has exactly two component, i.e. $G = G_1 \cup G_2$ and $r(G_1) \neq r(G_2)$, $\lambda(G_1) \neq \lambda(G_2)$ hold. Supposing $r(G_1) > r(G_2)$, we have that $\lambda(G_1) > \lambda(G_2)$.

2) Each component has the property

$$s(G_i) = r(G_i) - \lambda(G_i) \leq 4 \quad (i = 1, 2).$$

By Theorem 2 we conclude that each component is an induced subgraph of one of the graphs displayed in Fig. 2.

3) At least one of components has the largest eigenvalue greater than 2.

4) G_1 is one of the graphs $H_1 - H_5$ from Fig. 2, because they are only connected graphs which satisfy 2) and 3).

5) G_2 is a minimal graph w.r.t. the property $\lambda(G_2) < r(G_1) - 4$.

We distinguish the following five cases:

1° $G_1 = H_1$. Then G_2 is one of the graphs $C_4, C_6, P_7, V_{1,2}$ and S_4, because they are only graphs which satisfy conditions 2) and 5).

2° $G_1 = H_2$. Since H_1 is a proper induced subgraph of H_2, then G_2 satisfies 2), 5) and relation

$$\lambda(G_2) > r(H_1) - 4. \tag{2}$$

The graph P_6 is the unique graph satisfying all the above conditions.

3° $G_1 = H_3$. In this case H_1 is a proper induced subgraph of H_3, too, so G_2 must satisfy the conditions 2), 5) and (2). The graphs P_5 and S_3 are the unique such graphs.

4° $G_1 = H_4$. Then P_3 is the unique graph satisfying 2) and 5).

5° $G_1 = H_5$. Then G_2 is one of the graphs $C_4, C_6, C_8, P_9, V_{1,3}, V_{2,2}, W_2$ and S_4, since they are the unique graphs satisfying 2) and 5).

This completes the proof Theorem 3. □

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